The Effect of Privacy Regulations and Endogenous Marketing Research on Targeted Advertising[☆]

1st Paper (A Working Paper)

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Abstract

I develop a monopolistic competitive logit model where advertisers purchase noisy signals about consumers tastes through marketing research to explore how changing the cost of information affects consumer welfare, marketing research revenue, and advertiser platform revenue. Advertisers choose the number of noisy signals so that the marginal cost of an additional signal equals the marginal benefit of better targeting of their advertisements. I show that advertising and marketing research can be analyzed in a similar way to production inputs, and I find that the substitution between advertising and marketing research behaves similar to the classical findings on the substitution between production inputs. In addition, I find that marketing research costs have an ambiguous affect on consumer welfare. Higher marketing research costs decrease welfare by inducing a smaller, less-targeted selection of products. Yet higher marketing research costs increase welfare by inducing fewer annoying ads, even without any pricing effects.

Keywords: targeted advertising, marketing costs, mass advertising, privacy regulations, ad annoyance, information substitution

JEL Classification: M37, D83, M31, L51

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1. Introduction

Advertisers choose how much information they gather on consumers through marketing research. They do so by aggregating many independent, noisy signals about each consumer's tastes. Each additional signal gives them a more accurate aggregate signal on that consumer's taste. Advertisers use these signals to determine which consumers to target their ads. *Targeted advertising* is the sending of ads to a subset of consumers based on these signals. Yet each additional signal adds to the expense of doing more marketing research. Therefore advertisers choose the accuracy of this information, so that the marginal marketing research cost of an additional signal equals the marginal benefit of better targeting of their advertisements.

Consumers, including myself, appreciate advertisers being better informed about our tastes, because we get informed about a greater selection of more personalized products. I appreciate getting ads on products like DVDs and books, instead of ads on junk food or senior living, because I am more likely to buy DVDs and books. Therefore consumers benefit from advertisers getting more accurate aggregate signals about their tastes.

Despite this benefit, the Federal Trade Commission and the European Parliament have struggled with how to regulate targeted advertising to protect consumers' personal information. Goldfarb and Tucker (2011) showed that these privacy regulations in the European Union decreased the effectiveness of online ads. If better targeting of advertising is beneficial to consumers, then why would consumers want to protect their information from advertisers?

It could be that people value privacy because they want to protect against the criminal use of their information. Yet even with identity theft protection, regulation, and insurance, people feel uncomfortable with businesses knowing too much of their personal information. It could be that people value privacy because it contains some psychological benefit. This would make privacy a final good, instead of an intermediate. Yet Google has consumers opt out of their targeted advertising program, when the personal information used to target is stored on the consumer's personal computer. It could be that people value privacy because they want to avoid price discrimination. This possibility was explored when firms target advertise by Bergemann and Bonatti (2011).

Johnson (2013), the preliminary work of Shiman (1997), and this paper explore another explanation for why consumers value privacy with advertisement annoyance. When potential advertisers get too much information about consumers, too many advertisers enter the market, pestering consumers with too many annoying ads. If advertisers know that I like to read murder mystery novels, then I will be deluged with ads for murder mystery novels. Similarly if the viewers of a television program are very homogeneous, then they will be pestered with more advertisements. Personally, I dislike receiving junk mail and spam email more than I am worried about companies misusing my credit card numbers.[1] Especially, because my credit card provider protects me with account monitoring, password protection, and a guarantee to cover stolen funds.

[1] Don't tell the criminals.

In the previous literature on targeted advertising, advertisers get an exogenous noisy signal on consumer tastes. Privacy regulations directly increases the noise of this signal, therefore giving advertisers less information on consumers. In this paper, I explore how advertisers endogenously choose the accuracy of the information that they receive on consumers through marketing research. Privacy regulations indirectly affect the noise of the aggregate signal on a consumer's tastes by increasing the cost of gathering additional information (i.e. marketing research).

The cost of marketing research could be determined in a number of different ways, in addition to a government regulator choosing privacy regulations to protect consumer welfare. The costs of marketing research could be determined by the current level of technology. Therefore as technology improves, marketing research should get less costly, which should induce advertisers to gather more information on consumers. Another possibility is that the costs of marketing research are determined by marketing research firms. These marketing research firms set the prices of their information to maximize their own profits. In this paper, I explore the implications of these three different mechanisms for determining marketing research costs.

Marketing research is an input into the advertising decision of the firm. An advertiser chooses how much information to gather on consumers (the quantity of marketing research) and how many consumers to send its ad (the quantity of advertisements). Advertisements and marketing research are both complements in that marketing research increases the effectiveness of each ad. Yet they are both substitutes in that a firm could increase the number of ads it sends out to substitutes for doing less marketing research. This can be thought of as the substitution between the quality of targeting versus quantity of advertising, or it can be thought of as the substitution rate between information. I will refer to this as *information substitution*. In section 4.2, this will lead me to discover something similar to the Marginal Rate of Technical Substitution (MRTS) for marketing research and advertising, which I call the *Marginal Rate of Information Substitution* (MRIS).

Increasing the marginal cost of marketing research is similar to increasing the marginal cost of a production input. When the marginal cost of a production input increases, it not only increases a firm's own cost of production, it increases a firm's competitor's cost of production. Similarly, when the marginal cost of marketing research increases, it not only increases an advertiser's own cost of gathering information, it increases an advertiser's competitor's cost of gathering information. In this way, increasing the costs of marketing research could give advertisers profit and lead to market entry. This is because firm's are in a version of prisoner's dilemma where they would all be mutually better off advertising less but they are each individually better off advertising more. Increasing the marginal cost of marketing research reduces this negative externality.

2. Literature Review

This paper is most related to Johnson (2013). In Johnson (2013), improving the accuracy of the signal firms get on consumer tastes increases product selection and ad annoyance faced by consumers. In this paper, advertisers endogenously choose the accuracy of their signal firms on consumer tastes through marketing research.. Johnson (2013) further explores the impact of endogenous ad avoidance or ad blocking. In Johnson (2013), an improvement in signal accuracy encourages firms to advertise to more consumers, which encourages more consumers to block advertisements. Although in this paper, ad avoidance is exogenous. I consider the impact of endogenous ad avoidance at the end of the paper (to follow).

This paper is also related to Bergemann and Bonatti (2011). Both our papers analyze how improving the accuracy of the signal consumers get on consumer tastes improves consumer welfare through offering them a better selection of products. The biggest difference between our models is that Bergemann and Bonatti (2011) has no ad annoyance and the accuracy of the signal is exogenous. In his model, consumers are delivered a fixed number of ads. Firms buy these ads through perfect competition or from a monopolist publisher. This induces firms to advertise in fewer markets and increase the price of their products. While the product pricing effects are beyond the scope of this paper,[2] when I explore changing the price of advertising in the comparative statics sections of the paper (to follow), I show an additional effect on welfare through product selection (similar to Bergemann and Bonatti, 2011) and through ad annoyance. For a complete discussion on the product pricing effect, refer to Bergemann and Bonatti (2011).

This paper is also related to the preliminary work of Shiman (1997). Although Shiman abandoned the project, because there was very little interest at the time for targeted advertising in economics and the mathematics became too difficult, it is the first paper to explore how improving the accuracy of the signal firms get on consumer tastes increases product selection and ad annoyance faced by consumers. The biggest differences between our papers is in the purchasing decision and that I allow for endogenous marketing research. In Shiman (1997), a consumer's purchase decision is independent of what other products he bought. In this paper, I only allow a consumer to buy one product. In addition, I am able to explore advertisement prices, marketing costs, and the sharing of information between firms.

This paper is also related to Iyer et al. (2005). Both analyze firms decisions to either mass advertise or target advertise.[3] In Iyer et al. (2005), the cost of targeted advertising is some exogenous constant, and I allow for a marketing cost of gathering information (a better signal). Firms would pay for an equilibrium level of information, if they are using it to target, and not pay for any information, if they are mass advertising. In this way I am making the fixed targeting cost of Iyer et al. (2005) an endogenous constant.

Both Iyer et al. (2005) and this paper find that for a low per-consumer advertising cost or a high marketing cost, firms will mass advertise, and for a high per-consumer advertising cost or a low marketing cost, firms will target advertise.[4] I extend this to show that for a low

[2] there are plenty of papers that examine how consumer information can be used to price discriminate (see for example: Esteves, 2009; Goldfarb and Tucker, 2011; Villas-Boas, 1999, 2004), and several papers that extend this to targeted advertising (see for example: Galeotti and Moraga-González, 2004; Iyer et al., 2005).

[3] Iyer et al. (2005) analyzes the case of a duopoly, and I analyze the case of monopolistic competition.

[4] Iyer et al. (2005) also show that for an intermediate per-consumer advertising cost or an intermediate marketing cost, one firm will target advertise and the other will mass advertise. There results depend on the

signal accuracy, firms mass advertise, and for a high signal accuracy, firms mass advertise.

This paper is also related to Esteban et al. (2001). In their paper, the specialization of a magazine or other advertising medium is equivalent to signal accuracy. Consumers reading more specialized magazines are more likely to buy the product. Esteban et al. (2001) show that if a social planner picks product price and degree of advertisement specialization, then targeted advertising is preferable to mass advertising. I get a similar result. I find that if a social planner picks the marketing research cost, then targeted advertising is consumer welfare improving over mass advertising.

Esteban et al. (2001) also show that a firm might choose to specialize its advertising too much. This might make targeted advertising less desirable than mass advertising, because the firm may choose to increase its prices too much.[5] My paper extends this discussion by the addition of ad annoyance costs. I find that if firms are getting too accurate or too inaccurate information on consumers, then targeted advertising might be less desirable than mass advertising, because consumers face additional advertisement annoyance costs.

3. Model

There is a sufficiently large number N of profit maximizing firms. Each firm may potentially enter the market at an entry cost $F > 0$ and produce a product at a constant marginal cost normalized to zero. There is a unit mass of utility maximizing consumers. Half of the consumers are type 0 and the other half of the consumers are type 1. Initially, each firm only knows the aggregate distribution of types of consumers, but does not know the types of individual consumers nor does it have any initial signal on individual consumer types.

Each entrant j will choose the quantity z_j of signals to collect from each consumer, which I interpret as its marketing research. Each signal is a private, i.i.d. signal that is true with probability $\frac{1}{1+\phi}$ and false with a probability $\frac{\phi}{1+\phi}$, where $\phi \in [0, 1]$ is the error level of an individual signal. A $\phi = 0$ would be a perfect signal about consumers' types and a $\phi = 1$ would be a meaningless signal about consumers' types.

Entrant j pays a constant per signal marketing research cost of $m \geq 0$. Consumers and firms treat m as an exogenous constant. m could be many factors, including the level of privacy regulations, the tax or fee on information quality, and the efficiency of information technology. And it could be determined by a government regulator choosing privacy regulations, a marketing research firm choosing the price of its information, an ad platform that both sells ad space and sells marketing research, or even the current level of technology.

After choosing marketing research, each entrant j simultaneously chooses its product type $t_j \in \{0, 1\}$, which consumers to send its ad to, and its single product price p_j to maximize its own profit Π_j. For each consumer whom it chooses to advertise to, entrant j

fact that consumers who would be willing to buy either product (the comparison shoppers) have the same value r for both products. If both firms were to advertise to the comparison shoppers, then firms would both set a product price of zero in Bertrand style price competition.

[5] He gets this result when mass advertising wastes few ads and when product demand is sufficiently inelastic.

pays a constant per consumer advertising cost of c. If firm j enters, buys z_j signals from each consumer, advertises to v_j consumers, and sells q_j units of its good then it gets a profit of $\Pi_j \equiv p_j q_j - cv_j - mz_j - F$.

Consumers don't always see or remember every ad sent to them. Consumers see, remember, or retain ads with at an i.i.d. probability of $\alpha \in [0,1]$ (the ad retention probability). I will refer to those products whose ads are both sent to the consumer and retained by the consumer as the consumer's known products. These are the products that are available for the consumer to buy. And I will refer to those products whose ads are either not retained by the consumer or not sent to the consumer as the consumer's unknown products. These are the products that are unavailable for the consumer to buy.

Each consumer i may buy one unit of his choice from his known products, or he may choose to take an outside option.[6] The utility consumer i would get from buying good j is $u_{ij} \equiv \hat{u}_{ij} + \epsilon_{ij} - A(n)$, where $\hat{u}_{ij} \equiv R + b\lambda_{ij} - p_j$, $R > 0$ is the benefit from consuming any advertised good, $b > 0$ is the additional benefit from buying a good personalized to the consumer's type, λ_{ij} is one if consumer i's type $\tau_i \in \{0,1\}$ matches product j's type $t_j \in \{0,1\}$ and zero otherwise, p_j is the price of good j, ϵ_{ij} is an i.i.d. stochastic shock to consumer i's value for good j with a c.d.f. of $H(\epsilon) = e^{-e^{-(\frac{\epsilon}{\mu}+\gamma)}}$, γ is Euler's constant (≈ 0.5772),[7] $\mu > 0$ controls the variance of ϵ,[8] and $A(n)$ is the ad annoyance suffered by a consumer from retaining ads from n different firms. Consumers are not annoyed by those ads that are not retained. I assume that:

(A1) A is twice differentiable and continuous

(A2) $A'(n) > 0$ for all $n \geq 0$

Assumption (A1) rules out irregular special cases like jump discontinuities and allows us to use first and second order conditions. Assumption (A2) means consumers are always annoyed by more ads.

The utility consumer i would get from buying the outside option is $u_{i0} = \hat{u}_{i0} + \epsilon_{i0} - A(n)$, where $\hat{u}_{i0} = 0$ and ϵ_{i0} is an i.i.d. stochastic shock to consumer i's value for the outside option with a c.d.f. of $H(\epsilon)$.

4. Equilibrium

Consider consumer i of type $\tau_i \in \{0,1\}$. Let $\omega_{ij} = 1$ if product j is one of consumer i's known products and $\omega_{ij} = 0$ otherwise. And let $\omega_{i0} = 1$, because every consumer knows about the outside option. Then the probability that consumer i will buy product j is the

[6] This may simply be the option not to buy any good.

[7] The addition of the Euler's constant γ makes $E[\epsilon_{ij}] = 0$ and $H_\epsilon(0) = .5$.

[8] $VAR(\epsilon) = \frac{\pi^2}{6}\mu^2$.

well known multinomial logit result given by:[9]

$$\Pr_{\text{buy}}(i,j) = \omega_{ij}\frac{e^{\frac{R+\lambda_{ij}b-p_j}{\mu}}}{K_i} \tag{1}$$

where $K_i \equiv 1 + \sum_{h=1}^{N} \omega_{ih} e^{\frac{R+\lambda_{ih}b-p_h}{\mu}}$.

One way of thinking about K_i is that it is the price index for consumer i. A higher value of K_i means that known product j faces more competition, either more price competition or more competitors. In addition a higher value of K_i means that consumer i has more consumer surplus from the product that he buys, CS_i.[10]

Recall that consumer i only retains $\alpha \in [0,1]$ of the ads sent to him. Therefore if firm j sends consumer i its ad, then the probability that consumer i will retain the ad is α. Therefore, if firm j sends its ad to consumer i, then the probability of a sale is:

$$\Pr_{\text{sale}}(i,j) = \alpha\frac{e^{\frac{R+\lambda_{ij}b-p_j}{\mu}}}{K_i} \tag{2}$$

Firm j does not know consumer i's type, so it does not know λ_{ij} in equation (2). Recall that each entrant j does marketing research in the form of buying a quantity z_j of signals to collect from each consumer. Recall that each signal is a private, i.i.d. signal that is true with probability $\frac{1}{1+\phi}$ and false with a probability $\frac{\phi}{1+\phi}$. To simplify the language, let's call the signals that match the firm's type "matched signals" and the signals that do not match the firm's type "unmatched signals". Let y be the number of matched signals and let x be the total number of unmatched signals. Therefore for any given consumer i of type τ_i and entrant j of type t_j, the number of type $1-t_j$ signals x_{ij} is a binomial distribution (and therefore $y_{ij} = z_j - x_{ij}$), where:

$$\begin{aligned} \Pr(x_{ij} = k|\tau_i = t_j) &= \tbinom{z_j}{k}\left(\frac{1}{1+\phi}\right)^{z_j-k}\left(\frac{\phi}{1+\phi}\right)^{k} \\ \Pr(x_{ij} = k|\tau_i = 1-t_j) &= \tbinom{z_j}{k}\left(\frac{\phi}{1+\phi}\right)^{z_j-k}\left(\frac{1}{1+\phi}\right)^{k} \end{aligned} \tag{3}$$

Note that we cannot look at the limit case for large number of signals z_j. By the law of large numbers, when z_j is large, $x_{ij}/z_j = \frac{\phi}{1+\phi}$ if $\tau_i = t_j$ and $x_{ij}/z_j = \frac{1}{1+\phi}$ if $1-\tau_i = t_j$. Therefore for a large number of signals, the advertiser gets a perfect signal for each consumer's type.[11] For this reason, it would be inappropriate to approximate equation (3) with normal

[9] This result was first proved by Holman and Marley. See Luce and Suppes (1965) or Anderson et al. (1992).

[10] Anderson et al. (1992, p. 60-61) shows that this relation is $CS_i = \mu \ln K_i$.

[11] This is also a problem is you allow ϕ to approach 1 as z_j approached ∞. This limit case would avoid the problem with the law of large numbers, because $\lim_{\phi\to 1}\frac{\phi}{1+\phi} = \lim_{\phi\to 1}\frac{1}{1+\phi} = \frac{1}{2}$. Yet the variance of the

distributions using the de Moivre-Laplace theorem.

For notational simplicity let's define the function $g(z,x) \equiv \binom{z}{x}\left(\frac{1}{1+\phi}\right)^{z-x}\left(\frac{\phi}{1+\phi}\right)^{x}$. Then the conditional probability mass functions $\Pr(x_{ij}=k|\tau_i=t_j)=g(z_j,k)$ and $\Pr(x_{ij}=k|\tau_i=1-t_j)=g(z-k,z_j)=\theta_{ij}g(z,k)$, where $\theta_{ij}\equiv\phi^{z_j-2k}$. Therefore the probability mass function of x_{ij} is $\frac{1+\theta_{ij}}{2}g(z_j,x_{ij})$.

Note that the distribution of x_{ij} is a symmetric bimodal distribution with a mean and median of $x_{ij}=\frac{z_j}{2}$, and modes of $x_{ij}=z_j\frac{1}{1+\phi}$ and $x_{ij}=z_j\frac{\phi}{1+\phi}$. Therefore its probability mass function $\frac{1+\theta_{ij}}{2}g(z_j,x_{ij})$ is symmetric with a mean and median of $x_{ij}=\frac{z_j}{2}$, and modes of $x_{ij}=z_j\frac{1}{1+\phi}$ and $x_{ij}=z_j\frac{\phi}{1+\phi}$.

Applying Baye's Theorem, I find the probability of type $\tau_i\in\{0,1\}$ given θ_{ij}:

$$\Pr(\tau_i=t_j|\theta_{ij})=\frac{1}{1+\theta_{ij}} \quad (4)$$
$$\Pr(\tau_i=1-t_j|\theta_{ij})=\frac{\theta_{ij}}{1+\theta_{ij}}$$

The proofs of mathematical equations like this is in Appendix Appendix A which starts on page 29. I interpret θ_{ij} as entrant j's aggregate signal about consumer i's type. A $\theta_{ij}=0$ would mean that there is a 100% probability that consumer i is type t_j. A $\theta_{ij}=1$ would mean that the signal is meaningless; there is a 50% probability that consumer i is type t_j and a 50% probability that consumer i is type $1-t_j$. And a $\theta_{ij}=\infty$ would mean there is a 100% probability that consumer i is type $1-t_j$.

Therefore, if firm j sends its ad to consumer i, then the probability of a sale conditional on consumer i's signals is:

$$\Pr_{\text{sale}}(i,j|\theta_{ij})=\alpha[\frac{\frac{1}{1+\theta_{ij}}e^{\frac{R+b-p_j}{\mu}}}{K_{i,t_j}}+\frac{\frac{\theta_{ij}}{1+\theta_{ij}}e^{\frac{R-p_j}{\mu}}}{K_{i,1-t_j}}] \quad (5)$$

Firm j does not know consumer i's two possible price indices, $K_{i,0}$ or $K_{i,1}$, because the only information firm j knows about consumer i comes from firm j's private signal from its marketing research. Therefore firm j uses the expected value of the probabilities given by equation (5) to determine the probability of a sale. Therefore, if firm j sends consumer i its ad, the average probability that consumer i will buy product j conditional on consumer i's signals is:

$$\Pr_{\text{sale}}(i,j|\theta_{ij})=\alpha[\frac{\frac{1}{1+\theta_{ij}}e^{\frac{R+b-p_j}{\mu}}}{K_{t_j}}+\frac{\frac{\theta_{ij}}{1+\theta_{ij}}e^{\frac{R-p_j}{\mu}}}{K_{1-t_j}}] \quad (6)$$

where K_l is the harmonic mean[12] of the price indices of consumers of type $l\in\{0,1\}$. Because

probably distribution of x_{ij} would increase in z_j, so both distributions would approach each other.

[12] ${K_l}^{-1}=\int_0^1\frac{1/2}{(K_i)^{-1}}*1\{\tau_i=l\}d_i$, where τ_i is consumer i's type.

signals to a firm are independent, each firm faces the same K_1 and the same K_0. Therefore each firm treats K_1 and K_0 as market constants. I will later show in Lemma 3 that K_1 must equal K_0, but for now let's allow $K_1 \neq K_0$.

Note that equation (6) is independent of i and j. The only variables that matter are: the firm's price p_j, the firm's product type $t_j \in \{0, 1\}$, and its aggregate signal θ_{ij}. Therefore the marginal expected profit π for an entrant of type $t \in \{0, 1\}$ with a product price p for a consumer with an aggregate signal of θ is $\pi(p, t, \theta)$ given by:

$$\pi(p,t,\theta) = p * \alpha[\frac{\frac{1}{1+\theta} e^{\frac{R+b-p}{\mu}}}{K_t} + \frac{\frac{\theta}{1+\theta} e^{\frac{R-p}{\mu}}}{K_{1-t}}] - c \tag{7}$$

In this paper I analyze the case of monopolistic competition. Similar to the basic logit monopolistic competition model presented in Anderson et al. (1992, p. 221-226), this market structure is the limit case where there are so many firms that an individual firm's decisions do not impact the market variables K_1 and K_0, or an individual firm does not consider its impact on the the market variables K_1 and K_0 (as in Dixit and Stiglitz, 1977). Therefore firms treat K_1 and K_0 as exogenous constants. This market structure not only makes my model more tractable, but more realistically reflects many advertising markets, including online advertising, billboard advertising, and telemarketing.

Optimizing any firm j's profit over its price p_j, I find that for every entrant j:

$$p_j = \mu \tag{8}$$

Equation (8) finds that all entrants, independent of who they send their ads and how much marketing research, will always choose price $p = \mu$ to maximize their profits. This is not a limitation of the model, but a feature. In this paper, I examine the effects on the advertising side of the market. There are plenty of papers that examine product pricing effects of targeted advertising (see for example: Bergemann and Bonatti, 2011; Galeotti and Moraga-González, 2004; Iyer et al., 2005).

Substituting $p = \mu$ into equations (7), I find that the marginal expected profit π for an entrant of type t for a consumer with an aggregate signal of θ is $\pi(t, \theta)$ given by:

$$\pi(t,\theta) = \alpha\mu[\frac{\frac{1}{1+\theta} e^{\frac{R+b-\mu}{\mu}}}{K_t} + \frac{\frac{\theta}{1+\theta} e^{\frac{R-\mu}{\mu}}}{K_{1-t}}] - c = \frac{1}{1+\theta}\pi_t^t + \frac{\theta}{\theta+\theta}\pi_t^{1-t} \tag{9}$$

where $\pi_{t_j}^{\tau_i} \equiv \alpha\mu\frac{e^{\frac{R+b\lambda_{ij}-\mu}{\mu}}}{K_{\tau_i}} - c$ is firm j's additional expected profit from sending its ad to a consumer i. Note that firm j would treat π_0^0, π_0^1, π_1^0, and π_1^1 as market constants.

Lemma 1. *For every entrant j there exists one threshold $\overline{x}_j \in (0, 1]$, such that:*

(a) if $x_{ij} \leq \overline{x}_j$, then entrant j will send its ad to consumer i, and

(b) if $x_{ij} > \overline{x}_j$, then entrant j will not send its ad to consumer i.

Proof: Consider an arbitrary entrant with product type t. By equation (9), I have: $\frac{\partial}{\partial\theta}\pi(t,\theta) \gtreqless 0 \leftrightarrow \frac{-(\pi_t^t - \pi_t^{1-t})}{(1+\theta)^2} \gtreqless 0 \leftrightarrow \pi_t^{1-t} \gtreqless \pi_t^t$. Note that this does not depend on θ. I have three possibilities: (i) $\pi_t^{1-t} < \pi_t^t$, (ii) $\pi_t^{1-t} = \pi_t^t$, or (iii) $\pi_t^{1-t} > \pi_t^t$.

Note that an entrant must send ads to at least one consumer or else it would not choose to enter the market, due to the fixed entry cost $F > 0$. Therefore $x_j = 0$ is not possible.

Consider two arbitrary consumers, one who gives x type $1-t$ signals and one who gives x' type $1-t$ signals, where $x' < x$.

Further note that $\frac{\partial}{\partial x}\theta(x) = \frac{\partial}{\partial x}(\phi^{z_j - 2x}) = -2ln(\phi) * \phi^{z_j-2x} > 0$. Therefore $\theta(x') < \theta(x)$.

Case i: $\pi_t^{1-t} < \pi_t^t \leftrightarrow \frac{\partial}{\partial\theta}\pi(t,\theta) < 0$:

Then by equation (9), if $\pi(t,\theta(x)) > 0$ then $\pi(t,\theta(x')) > 0$. Because the choice of x and $x' < x$ is arbitrary, if $\pi(t,\theta(x)) > 0$ then $\pi(t,\theta(x')) > 0$ for any $x' < x$. Therefore if firm j enters, there exist one threshold $\overline{x}_j$ that satisfies the lemma.

Case ii: $\pi_t^{1-t} = \pi_t^t \leftrightarrow \frac{\partial}{\partial\theta}\pi(t,\theta) = 0$:

Therefore the expected profit from sending an ad to any consumer is the same, independant of the signal. Because the firm must make profit on at least one consumer to enter, it must make a profit on all consumers. Therefore we have the lemma with $\overline{x}_j = z_j$.

Case iii: $\pi_t^{1-t} > \pi_t^t \leftrightarrow \frac{\partial}{\partial\theta}\pi(t,\theta) > 0$:

Then by equation (9), if $\pi(t,\theta(x')) > 0$ then $\pi(t,\theta(x)) > 0$. Because the choice of x and $x' < x$ is arbitrary, if $\pi(t,\theta(x')) > 0$ then $\pi(t,\theta(x)) > 0$ for any $x' < x$. Therefore if firm j enters, there exist one threshold $\overline{x}_j$ such that if $x_{ij} \geq \overline{x}_j$, then entrant j will send its ad to consumer i, and if $x_{ij} < \overline{x}_j$, then entrant j will not send its ad to consumer i.

Note that because $\pi_t^{1-t} > \pi_t^t \leftrightarrow \alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} - c > \alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_t} - c \leftrightarrow \frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} > \frac{e^{\frac{R+b-\mu}{\mu}}}{K_t}. \leftrightarrow \frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} > \frac{e^{\frac{R-\mu}{\mu}}}{K_t}$.

For all $x \in [z_j/2, z_j]$, I have that $\theta(x) \geq 1$ and consequentially $\frac{1}{1+\theta(x)} \leq \frac{\theta(x)}{1+\theta(x)}$ $\leftrightarrow \frac{1}{1+\theta(x)}\frac{e^{\frac{R-\mu}{\mu}}}{K_t} < \frac{\theta(x)}{1+\theta(x)}\frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} \leftrightarrow \frac{\frac{1}{1+\theta(x)}e^{\frac{R+b-\mu}{\mu}}}{K_t} + \frac{\frac{\theta(x)}{1+\theta(x)}e^{\frac{R-\mu}{\mu}}}{K_{1-t}} < \frac{\frac{1}{1+\theta(x)}e^{\frac{R-\mu}{\mu}}}{K_t} + \frac{\frac{\theta(x)}{1+\theta(x)}e^{\frac{R+b-\mu}{\mu}}}{K_{1-t}} \leftrightarrow$ $\pi(t,\theta) < \pi(1-t, 1-\theta)$, which is the profit entrant j would get as type $1-t$ sending the ad to the same consumer. Therefore for each and every consumer $x \in [z_j/2, z_j]$, firm j would make more profit as a type $1-t$ firm, instead of as a type t firm.

If $\overline{x}_j \geq z_j/2$, then entrant j would make more profit as type $1-t$ entrant sending its ad to the same set of consumers. This contradicts the assumption that entrant j is a type t firm, so let us assume that $\overline{x}_j < z_j/2$.

Because the probability mass function is symmetric about $z_j/2$, for each consumer $x_1 \in [\overline{x}_j, z_j/2)$ there exists a consumer $x_2 = z_j - x_1 \in (z_j/2, z_j - \overline{x}_j]$, and for each consumer $x_2 \in (z_j/2, z_j - \overline{x}_j]$ there exists a consumer $x_1 = z_j - x_2 \in [\overline{x}_j, z_j/2)$. Also by equation (9), I have the symmetry $\pi(t,\theta(x_1)) + \pi(t,\theta(x_2)) = \pi(1-t, 1-\theta(x_1)) + \pi(1-$

$t, 1-\theta(x_2))$, which means that entrant j would make the same expected profit on all the consumers $x \in [\overline{x}_j, z_j - \overline{x}_j]$ as a type $1-t$ firm. Because I have already shown that entrant j would make a higher expected profit on all the consumers $x \in (z_j - \overline{x}_j, x_j]$, I have that entrant j's total expected profit would be higher as a type $1-t$ firm. This contradicts the assumption that entrant j is a type t firm, which proves the lemma. □

Lemma 1 shows that any entrant will always send ads to the consumers with the least unmatched signals. In order to enter the market as type t: (1) the marginal profit from sending ads to the consumers with the least unmatched signals has to be positive, and (2) it has to be weakly more profitable to be type t, which rules out strategies were the firm targets consumers with the most unmatched signals.

Therefore the total expected profit Π for an entrant of type $t \in \{0, 1\}$ that pays for z signals and sends ads to each consumer with no more than $\overline{x}$ unmatched signals is:

$$\Pi(t, z, \overline{x}) = \sum_{x=0}^{\overline{x}} \pi(t, \theta)[\frac{1+\theta}{2} g(z, x)] - F - mz \tag{10}$$

where $\overline{\theta} = \phi^{z-\overline{x}}$ and $\frac{1+\overline{\theta}}{2} g(z, x)$ is the probability mass function of x.

In order to simplify equation (10), I introduce notation for the cumulative density function for equation (3), which is given by:

$$G_\rho(z, k) \equiv \frac{1}{2} \sum_{x=0}^{k} \rho^{z-2x} * \binom{z}{x} \left(\frac{1}{1+\phi}\right)^{z-x} \left(\frac{\phi}{1+\phi}\right)^{x} \tag{11}$$

where $\Pr(x_{ij} = x | \tau_i = t_j)$ and $\Pr(x_{ij} = x | \tau_i = 1-t_j)$ are given by equation (3). Therefore the conditional probability density functions are $G_1(z, k) = \Pr(x_{ij} \leq k | \tau_i = t_j)$ and $G_\phi(z, k) = 1 - G_1(z, z-k-1) = \Pr(x_{ij} \leq k | \tau_i = 1 - t_j)$.[13]

Note that the total profit function for entrant j can be rewritten as:

$$\Pi(t, z, \overline{x}) = \frac{1}{2} G_1(z, \overline{x}) \pi_t^t + \frac{1}{2} G_\phi(z, \overline{x}) \pi_t^{1-t} - F - mz \tag{12}$$

Because it is easier to interpret the number of consumers sent ads v that a firm sends than the maximum number of unmatched signals $\overline{x}$, I introduce a transformation of equation (12). Because half of the consumers are type t and the other half of the consumers are type $1-t$, I have that:

$$v = \frac{G_1(z, x) + G_\phi(z, x)}{2} \tag{13}$$

Consider how v relates to $\overline{x}$. The $\frac{\partial v}{\partial x}$ would be total additional consumers sent ads from increasing the maximum number of unmatched signals from $\overline{x} - 1$ to $\overline{x}$. Therefore it is

[13] G_ρ can also be represented in terms of the regularized incomplete beta function, which is included in many statistical packages, including Stata and Matlab.

the number of consumers with exactly $\overline{x}$ unmatched signals. There are $\frac{1}{2}g(z,\overline{x})$ matched consumers with exactly $\overline{x}$ unmatched signals. And there are $\frac{1}{2}\overline{\theta}g(z,\overline{x})$ unmatched consumers with exactly $\overline{x}$ unmatched signals. Therefore I have:

$$\frac{\partial v}{\partial \overline{x}} = \frac{1+\overline{\theta}}{2}g(z,\overline{x}) > 0 \tag{14}$$

Consider increasing z to $z+1$, while holding $\overline{x}$ constant. If a consumer already has more than $\overline{x}$ unmatched signals, then with an additional signal, it will still have more than $\overline{x}$ unmatched signals. If a consumer already has less than $\overline{x}$ unmatched signals, then an additional signal will not give it more than $\overline{x}$ unmatched signals. So an additional signal will only screen out those consumers who currently have $\overline{x}$ unmatched signals. Because there are $\frac{1}{2}g(z,\overline{x})$ matched consumers, and because matched consumers will give an additional unmatched signal with a probability of $\frac{\phi}{1+\phi}$, the advertiser would lose $\frac{1}{2}\frac{\phi}{1+\phi}g(z,\overline{x})$ matched consumers. Because there are $\frac{1}{2}\overline{\theta}g(z,\overline{x})$ unmatched consumers, and because unmatched consumers will give an additional unmatched signal with a probability of $\frac{1}{1+\phi}$, the advertiser would lose $\frac{1}{2}\frac{1}{1+\phi}\overline{\theta}g(z,\overline{x})$ matched consumers. Therefore the change in the total number of consumers sent ads is:

$$v(z+1,\overline{x}) - v(z,\overline{x}) = -\frac{1}{2}\frac{\overline{\theta}+\phi}{1+\phi}g(z,x) < 0 \tag{15}$$

Therefore v is strictly increasing in $\overline{x}$ and strictly decreasing in z. Therefore we can invert equation (13) to solve for $\overline{x}(z,v)$. I have that $\overline{x}(z,v)$ is strictly increasing in v. In order to increase z, holding v constant, I must have $0 = \frac{dv}{d\overline{x}}d\overline{x} + \frac{dv}{dz}dz$. This becomes:

$$\frac{d\overline{x}}{dz} = \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})} > 0 \tag{16}$$

Therefore $\overline{x}(z,v)$ is also strictly increasing in z. Therefore I transform equation (12) to:

$$\Pi(t,z,v) = v[\frac{1}{1+\theta^*(z,v)}\pi_t^t + \frac{\theta^*(z,v)}{1+\theta^*(z,v)}\pi_t^{1-t}] - F - mz \tag{17}$$

where $\theta^*(z,v) \equiv \frac{G_\phi(x(z,v),z)}{G_1(x(z,v),z)}$.

In order to rule out empty market or a monopoly equilibria, I assume that:

$$\frac{\mu}{2(F+c)} - \frac{1}{\alpha[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]} > 1 \tag{18}$$

which is sufficient to show that at least two firms enter the market.

Price competition will make at least one firm enter as type 0 and one firm enter as type 1, which I show in Lemma 2.

Lemma 2. *At least one firm enters as each product type.*

Proof: Suppose not. Then by equation (18), I have that there are two firms of the same type. Let us call this type, type $t \in \{0, 1\}$, and the two firms, firms 0 and 1. Note that both (z_0, v_0) and (z_1, v_1) must be optimal choices of (z, v) for $\Pi(t, z, v)$, but (z_0, v_0) is not necessarily equal to (z_1, v_1).

$K_t = 1 + 2\alpha[\frac{1}{1+\theta^*(z_0,v_0)}v_0 + \frac{1}{1+\theta^*(z_1,v_1)}v_1]e^{\frac{R+b-\mu}{\mu}}$

$> 1 + 2\alpha[\frac{\theta^*(v_0,z_0)}{1+\theta^*(z_0,v_0)}v_0 + \frac{\theta^*(z_1,v_1)}{1+\theta^*(z_1,v_1)}v_1]e^{\frac{R-\mu}{\mu}} = K_{1-t}$.

Because $\frac{1}{1+\theta^*(z_0,v_0)} > \frac{\theta^*(z_0,v_0)}{1+\theta^*(z_0,v_0)}$, I have:

$\Pi(t, z_0, \overline{x}_0) = v_0[\frac{1}{1+\theta^*(z_0,v_0)}\pi_t^t + \frac{\theta^*(z_0,v_0)}{1+\theta^*(z_0,v_0)}\pi_{1-t}^t] - F - mz_0$

$= v_0[\frac{1}{1+\theta^*(z_0,v_0)}(\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_t} - c) + \frac{\theta^*(z_0,v_0)}{1+\theta^*(z_0,v_0)}(\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} - c)] - F - mz_0$

$< v_0[\frac{1}{1+\theta^*(z_0,v_0)}(\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_{1-t}} - c) + \frac{\theta^*(z_0,v_0)}{1+\theta^*(z_0,v_0)}(\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_t} - c)] - F - mz_0 = \Pi(1-t, z_0, v_0)$

Therefore it is more profitable for firm 0 to be type $1 - t$, which means all firms can't be type t. □

Lemma 2 rules out empty product types. Therefore by free entry, the profit from being a type 0 and a type 1 firm should be equal. This ensures that the price indices $K_0 = K_1$, given by:

Lemma 3. $K_0 = K_1$

Proof: For any arbitrary type $t \in \{0, 1\}$, let (z_t, v_t) denote any optimal choices of (z, v) for $\Pi(t, z, v)$. From Lemma 2, I have that there is at least one type t firm and one type $1-t$ firm. Because of free entry, I have $\Pi(t, z_t, v_t) = \Pi(t, z_{1-t}, v_{1-t})$. Because (z_{1-t}, v_{1-t}) is an optimal choices of (z, v) for $\Pi(1-t, z, v)$, I have $\Pi(1-t, z_{1-t}, v_{1-t}) \geq \Pi(1-t, z_t, v_t)$. Therefore $\Pi(t, z_t, v_t) \geq \Pi(1-t, z_t, v_t)$

$\leftrightarrow v_t[\frac{1}{1+\theta^*(z_t,v_t)}\pi_t^t + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}\pi_t^{1-t}] - F - mz_t \geq v_t[\frac{1}{1+\theta^*(z_t,v_t)}\pi_{1-t}^t + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}\pi_{1-t}^{1-t}] - F - mz_t$

$\leftrightarrow v_t[\frac{1}{1+\theta^*(z_t,v_t)}(\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_t} - c) + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}(\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_t} - c)] - F - mz_t$

$\geq v_t[\frac{1}{1+\theta^*(z_t,v_t)}(\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_{1-t}} - c) + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}(\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} - c)] - F - mz_t$

$\leftrightarrow \frac{1}{1+\theta^*(z_t,v_t)}\frac{e^{\frac{R+b-\mu}{\mu}}}{K_t} + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}\frac{e^{\frac{R-\mu}{\mu}}}{K_{1-t}} \geq \frac{1}{1+\theta^*(z_t,v_t)}\frac{e^{\frac{R+b-\mu}{\mu}}}{K_{1-t}} + \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}\frac{e^{\frac{R-\mu}{\mu}}}{K_t}$

$\leftrightarrow K_t \leq K_{1-t}$ (because $\frac{1}{1+\theta^*(z_t,v_t)} \geq \frac{\theta^*(z_t,v_t)}{1+\theta^*(z_t,v_t)}$ and $e^{\frac{R+b-\mu}{\mu}} > e^{\frac{R-\mu}{\mu}}$).

Therefore for any arbitrary type $t \in \{0, 1\}$, $K_t \leq K_{1-t}$. Choosing type $t = 0$, I have $K_0 \leq K_1$. Choosing type $t = 1$, I have $K_1 \leq K_0$. Therefore $K_0 = K_1$. □

Therefore there is one common price index for both sub-markets, which I define to be $K \equiv K_0 = K_1$. Because competing firms only influence each other through the price index K, see equations (7) and (10), entrants face symmetric pricing and marketing research choices on either side of the market. Later I will show that there is one unique symmetric equilibrium, but for now let's entertain the possibility of multiple equilibria and asymmetric equilibria.

Therefore I define the marginal profit for a consumer with a signal of θ as $\pi(\theta) \equiv \pi(t,\theta) = \pi(1-t,\theta)$. I define the marginal profit for a consumer that matches the firm's type as $\pi_{\mathrm{M}} \equiv \pi^t_t = \pi^{1-t}_{1-t}$. I define the marginal profit for a consumer that does not match the firm's type as $\pi_{\mathrm{U}} \equiv \pi^{1-t}_t = \pi^t_{1-t}$. And I define profit as $\Pi(z,v) \equiv \Pi(t,z,v) = \Pi(1-t,z,v)$.

If there is positive profit, then more firms will enter, until profit is zero. If there is negative profit, then firms will leave, until profit is zero. This gives us a zero-profit condition. Using similar reasoning to the proof of Lemma 3 and this zero-profit condition, I rule out the possibility of multiple potential price indices:

Lemma 4. *There is one unique potential K.*

Proof: Let K^0 and K^1 be any two arbitrary price indices. For any arbitrary $\ell \in \{0,1\}$, let (z^ℓ, v^ℓ) denote any optimal choices of (z,v) under price index $\ell \in \{0,1\}$, and let $\Pi^\ell(z,v)$ denote a firm's profit under price index $\ell \in \{0,1\}$. Because of the zero profit condition, I have that $\Pi^\ell(z^\ell,v^\ell) = \Pi^{1-\ell}(z^{1-\ell},v^{1-\ell}) = 0$. Because $(z^{1-\ell},v^{1-\ell})$ is an optimal choices of (z,v) for $\Pi^{1-\ell}(z,v)$, I have $\Pi^{1-\ell}(z^{1-\ell},v^{1-\ell}) \geq \Pi^{1-\ell}(z^\ell,v^\ell)$.
Therefore $\Pi^\ell(z^\ell,v^\ell) \geq \Pi^{1-\ell}(z^\ell,v^\ell)$

$$\leftrightarrow v^\ell[\tfrac{1}{1+\theta^*(z^\ell,v^\ell)}(\alpha\mu\tfrac{e^{\frac{R+b-\mu}{\mu}}}{K^\ell}-c)+\tfrac{\theta^*(z^\ell,v^\ell)}{1+\theta^*(z^\ell,v^\ell)}(\alpha\mu\tfrac{e^{\frac{R-\mu}{\mu}}}{K^\ell}-c)]-F-mz^\ell$$
$$\geq v^\ell[\tfrac{1}{1+\theta^*(z^\ell,v^\ell)}(\alpha\mu\tfrac{e^{\frac{R+b-\mu}{\mu}}}{K^{1-\ell}}-c)+\tfrac{\theta^*(z^\ell,v^\ell)}{1+\theta^*(z^\ell,v^\ell)}(\alpha\mu\tfrac{e^{\frac{R-\mu}{\mu}}}{K^{1-\ell}}-c)]-F-mz^\ell$$
$$\leftrightarrow \tfrac{1}{1+\theta^*(z^\ell,v^\ell)}\tfrac{e^{\frac{R+b-\mu}{\mu}}}{K^\ell}+\tfrac{\theta^*(z^\ell,v^\ell)}{1+\theta^*(z^\ell,v^\ell)}\tfrac{e^{\frac{R-\mu}{\mu}}}{K^\ell} \geq \tfrac{1}{1+\theta^*(z^\ell,v^\ell)}\tfrac{e^{\frac{R+b-\mu}{\mu}}}{K^{1-\ell}}+\tfrac{\theta^*(z^\ell,v^\ell)}{1+\theta^*(z^\ell,v^\ell)}\tfrac{e^{\frac{R-\mu}{\mu}}}{K^{1-\ell}} \leftrightarrow K^\ell \leq K^{1-\ell}$$

Therefore for any arbitrary price index $\ell \in \{0,1\}$, $K^\ell \leq K^{1-\ell}$. Choosing price index $\ell = 0$, I have $K^0 \leq K^1$. Choosing price index $\ell = 1$, I have $K^1 \leq K^0$. Therefore $K^0 = K^1$. □

Next I will consider the three possible cases: all entrants do no marketing research (section 4.1), all entrants do marketing research (section 4.2), and some entrants do marketing research while other entrants do no marketing research (section 4.3).

4.1. Mass Advertising

In this section, I consider the case where all entrants do no marketing research. In order for a firm to enter, the expected marginal profit π needs to be positive for at least one consumer. If an entrant does no marketing research, then its expected marginal profit from each consumer is identical, and therefore positive. Therefore I have that all firms *mass advertise*, which I define as:

Definition 1. mass advertise: to send an ad to all consumers.

Therefore each entrant will get the *mass advertising profit* given by:

$$\Pi_{MA} \equiv \Pi(z=0, v=0) = \frac{1}{2}\pi_{\mathrm{M}} + \frac{1}{2}\pi_{\mathrm{U}} - F \tag{19}$$

Note that mass advertising is equivalent to $v = 1$ and $z = 0$.

Let n_0 be the number of entrants of type 0 and n_1 be the number of entrants of type 1. Then each entrant j would make the expected marginal profit of $\pi = \frac{1}{2}[\pi_{\mathrm{M}} + \pi_{\mathrm{U}}]$ on each consumer. Therefore by free entry, I have the zero profit condition $F = \pi$, which solving for n_0 and n_1 becomes:

$$n_{MA} \equiv n_0 = n_1 = \frac{\mu}{2(F+c)} - \frac{1}{\alpha[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]} \tag{20}$$

Note that (20) simplifies equation (18) to $N_{MA} > 1$. Therefore I am assuming that the market can at least sustain a mass advertising equilibrium.

Here $\frac{\mu}{2(F+c)}$ is the number of firms that would enter each sub-market if there were no outside option. It is also the effect of the entry cost F and the advertising cost c on the entry. And $[\alpha(e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}})]^{-1}$ is the number of firms discouraged from entering each sub-market due to the outside option. Using (20) to solve for K, I have:

$$K_{MA} = \frac{\alpha\mu}{2(F+c)}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] \tag{21}$$

Therefore I have that marketing research costs would have no marginal effect on the price index when all firms are not doing any marketing research.

Lemma 5. $K \geq K_{MA}$

Proof: Suppose by way of contradiction that $K_{MA} > K \leftrightarrow \frac{\alpha\mu}{2(F+c)}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] > K$
$\leftrightarrow \frac{\alpha\mu}{2}\frac{e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}}{K} - F - c > 0$
Therefore entrants could make a positive profit from mass advertising. Therefore all entrants would choose to mass advertise, which would mean that $K_{MA} = K$. □

If $\pi_U > 0$, then an entrant would send its ads to every consumer, because it expects a marginal profit on each ad it sends out. Therefore I assume: $\pi_U < 0$, which is equivalent to:

$$\frac{F}{c} < \frac{1}{2}(e^{b/\mu} - 1) \tag{22}$$

4.2. Targeted Advertising

In this section, I consider the case where all entrants do marketing research. When a firm does any marketing research (i.e. $z > 0$), then it must be sending its ads to some but not all consumers (i.e. $0 < v < 1$). Otherwise it would be more profitable not to do any marketing research. Therefore by Lemma 1, I have that all entrants *target advertise*, which I define as:

Definition 2. target advertise: to send an ad to all and to only those consumers with no more than $\overline{x}$ unmatched signals, where $0 \leq \overline{x} < z$.

Note that by definition, if a firm is target advertising, then $z > 0$.

If the firm sends one more ad v then it is sending an additional ad to a threshold consumer, who has a $\theta = \overline{\theta}$. Therefore with a probability of $\frac{1}{1+\overline{\theta}}$ it will send an ad to a matched consumer, each of whom yields an expected marginal revenue of $\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K}$. And with a probability of $\frac{\overline{\theta}}{1+\overline{\theta}}$ it will send an ad to a unmatched consumer, each of whom yields an expected marginal revenue of $\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K}$. Therefore the expected marginal revenue from sending an additional ad is:

$$MR_v = \alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} > 0 \tag{23}$$

If the firm sends one more signal without changing $\overline{x}$, the maximum number of unmatched signals, then it will be advertising to fewer consumers. If a consumer already has more than $\overline{x}$ unmatched signals, then with an additional signal, it will still have more than $\overline{x}$ unmatched signals. If a consumer already has less than $\overline{x}$ unmatched signals, then an additional signal will not give it more than $\overline{x}$ unmatched signals. So an additional signal will only screen out those consumers who currently have $\overline{x}$ unmatched signals.

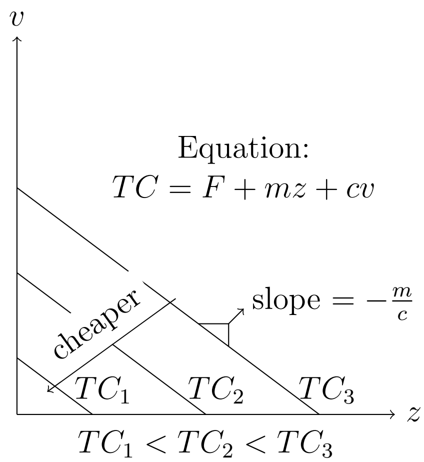
There are $\frac{1}{2}g(z,\overline{x})$ matched consumers that currently have a signal of $\overline{x}$, each of whom yields an expected marginal revenue of $\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K}$. An additional signal would mean that the firm will not send an at to $\frac{\phi}{1+\phi}$ of these matched consumers. And there are $\frac{1}{2}\overline{\theta}g(z,\overline{x})$ unmatched consumers that currently have a signal of $\overline{x}$, each of whom yields an expected marginal revenue of $\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K}$. An additional signal will mean that the firm will not send an at to $\frac{1}{1+\phi}$ of these unmatched consumers. Therefore if the firm sends out an additional signal without changing $\overline{x}$, it will send out $\frac{1}{2}[\frac{\phi}{1+\phi} + \frac{1}{1+\phi}\overline{\theta}]g(z,\overline{x}(v,z))$ fewer ads and lose the revenue of $\frac{\alpha\mu}{2}g(z,\overline{x}(z,v))\frac{\frac{\phi}{1+\phi}e^{\frac{R+b-\mu}{\mu}} + \frac{1}{1+\phi}\overline{\theta}e^{\frac{R-\mu}{\mu}}}{K}$ from sales.

Note that having the firm send out one more signal without changing the number of ads can be split into two steps: (Step 1) the firm sends out an additional signal, while holding $\overline{x}$ constant. This causes the firm to advertise to $\frac{1}{2}[\frac{\phi}{1+\phi} + \frac{1}{1+\phi}\overline{\theta}]g(z,\overline{x}(v,z))$ fewer consumers. (Step 2) the firm sends advertises to an additional $\frac{1}{2}[\frac{\phi}{1+\phi} + \frac{1}{1+\phi}\overline{\theta}]g(z,\overline{x}(v,z))$ consumers. These consumers would be threshold consumers, so it would earn a marginal revenue of MR_v for each of these additional consumers. Therefore the marginal revenue from sending an additional signal is:

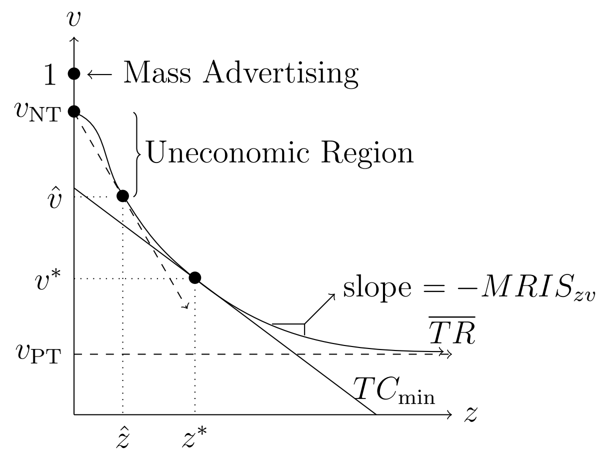
$$\begin{aligned} MR_z &= -\frac{\alpha\mu}{2}g(z,\overline{x}(z,v))\frac{\frac{\phi}{1+\phi}e^{\frac{R+b-\mu}{\mu}} + \frac{1}{1+\phi}\overline{\theta}e^{\frac{R-\mu}{\mu}}}{K} + \frac{1}{2}[\frac{\phi}{1+\phi} + \frac{1}{1+\phi}\overline{\theta}]g(z,\overline{x}(v,z))MR_v \\ &= \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}g(z,\overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K} > 0 \end{aligned} \tag{24}$$

When an advertiser gathers an additional signal about consumers tastes, then each advertisement is more likely to reach a matched consumer and less likely to reach an unmatched

Figure 1: Cost Minimization Problem



(a) Iso-Cost Lines



(b) Iso-Revenue Curve

consumer. Therefore the advertiser does not have to send out as many ads to get the same sales revenue. This substitution between marketing research and advertising is given by the **marginal rate of information substitution** ($MRIS_{zv}$), which I define as the number of additional ads v needed in order to retain the same Total Revenue if the firm gathers one fewer signal z about consumers tastes.

$$MRIS_{zv} = \frac{MR_z}{MR_v} = \frac{1}{2}\frac{1-\phi}{1+\phi}\overline{\theta}g(z,\overline{x}(z,v))\frac{e^{b/\mu}-1}{e^{b/\mu}+\overline{\theta}} > 0 \tag{25}$$

$MRIS_{zv}$ is always positive, because the firm would always need to do more advertising to get the same Total Revenue with less marketing research. The $MRIS_{zv}$ can be thought of as the substitution rate between the quality of targeting versus quantity of advertising, or it can be thought of as the substitution rate between information. It is the rate of how many more consumers does an advertiser need to inform about its product if it is less informed about consumers' tastes.

Figure 1 illustrates the cost minimization problem of a firm that is targeted advertising.

Figure 1 (1a) illustrates the cost of marketing research z (on the horizontal axis) and advertising v (on the vertical axis). It shows three iso-cost lines, TC_1, TC_2, and TC_3, which are the combinations of marketing research z and advertising v that cost the same. I have that each iso-cost line has a slope of $-\frac{m}{c}$. And I have that a closer iso-cost line to the origin represents a cheaper Total Cost. Therefore $TR_1 < TR_2 < TR_3$.

Figure 1 (1b) illustrates an iso-revenue curve, which is the combination of marketing research z and advertising v that yield the same Total Revenue. The slope of the iso-revenue curve would be $-MRIS_{zv}$.

The cost minimizing level of marketing research z and advertising v would be the point on the iso-revenue curve that is on the lowest possible iso-cost line. This happens when the iso-cost line is tangent to the iso-revenue curve. Therefore the cost-minimizing condition would be $MRIS_{zv} = \frac{m}{c}$. Or equivalently that $\frac{MR_z}{m}$, which is the marginal revenue per dollar spent on marketing research, is equal to $\frac{MR_v}{c}$, which is the marginal revenue per dollar spent on advertising.

I will show in Lemma 6 that $MRIS_{zv}$ is not always decreasing. When $MRIS_{zv}$ is increasing, I have that the iso-revenue curve is concave, and therefore $MRIS_{zv} = \frac{m}{c}$ would mean that the firm would be cost maximizing. I will show in Lemma 6 that the potential profit minimizing (and therefore cost maximizing) point occurs when $\overline{x} < \frac{\phi}{1+\phi}$, and I will show that the profit maximizing (and therefore cost minimizing) point occurs when $\overline{x} > \frac{\phi}{1+\phi}$. This creates the potential uneconomic region shown in 1 (1b).

I will next solve for the profit maximizing level of advertising v and marketing research z. I will start by solving for the zero-profit condition. Each entrant j would make the expected marginal profit of π_{M} on each of the $\frac{1}{1+\theta^*(z,v)}v$ consumers that it advertises to who match its type and π_{U} on each of the $\frac{\theta^*(z,v)}{1+\theta^*(z,v)}v$ consumers that it advertises to who do not match its type. Therefore by free entry, I have the zero-profit condition:

$$F + mz = v[\frac{1}{1+\theta^*(z,v)}\pi_{\text{M}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)}\pi_{\text{U}}] \tag{26}$$

If a firm is targeted advertising, then it is optimizing over its choice of v, the total number ads. The profit maximizing level of advertising v would occur when the marginal revenue MR_v from advertising is equal to the marginal cost c of advertising. This is equivalent to the marginal profit $\pi(\overline{\theta}) = 0$. If the marginal profit $\pi(\overline{\theta}) > 0$, then it would be profitable for the firm to increase v. If the marginal profit $\pi(\overline{\theta}) < 0$, then it would be profitable for the firm to decrease v. Therefore the first-order condition for v is given by:

$$\begin{aligned} &\pi(\overline{\theta}) = 0 \\ &\leftrightarrow \overline{\theta} = \frac{\pi_{\text{M}}}{-\pi_{\text{U}}} > 0 \end{aligned} \tag{27}$$

where $\overline{\theta} \equiv \overline{\theta}_t = \overline{\theta}_{1-t}$.

This defines a unique v^*, because $\overline{\theta}(z,v)$ is strictly increasing in v. Note that equation (22) guarantees $\pi_{\text{U}} < 0$, and that I must have $\pi_{\text{M}} > 0$ for any firm to enter the market. Therefore $\overline{\theta} > 0$.

Likewise, if a firm is targeted advertising, then it is optimizing over its choice of z, the total number of signals. The profit maximizing level of marketing research z would occur when the marginal revenue MR_z from marketing research is equal to the marginal cost m of marketing research. Therefore the first-order condition for z is given by:

$$m = \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}g(z,\overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K} \tag{28a}$$

$$\leftrightarrow m = \frac{1}{2}(\frac{1-\phi}{1+\phi})g(z,\overline{x}(z,v))\pi_{\text{M}} \tag{28b}$$

where z^* is the optimum z.

Note that there are potentially two solutions to equation (28b), because the distribution of $g(x,z)$ is inverse-u-shaped in z (think the probability mass function of a negative binomial distribution). The lower of the two potential values would be the the z that minimizes profit. And the higher of the two potential values would be the the z that maximizes profit. Therefore the advertiser would choose the higher of the two values. It can be shown that equation (28a) only has one potential solution, which is equal to the higher of the two values of z that solves equation (28b).

This alone does not guarantee that the combination of z^* and v^* is unique, because equations (27) and (28b) both depend on z and v. Therefore to rule out the possibility of saddle points, I use the fact that $\overline{\theta}$ is fixed to prove that the combination of z^* and v^* is unique:

Lemma 6. *For a given K, there is at most one optimal (z^*, v^*).*

Proof: Because K is a unique constant (Lemma 4), I have that $\frac{\pi_M}{-\pi_U}$ is a unique constant. Therefore, by equation (27), I have that $\overline{\theta} = \phi^{z-2\overline{x}(z,v)}$ is a unique constant. Therefore I consider increasing z, while holding $\overline{\theta}$ (or equivalently $z - 2\overline{x}$) constant.

$\leftrightarrow \frac{d\Pi}{dz} = [MR_z - m] + \frac{\overline{x}}{dz}\frac{\partial v}{\partial \overline{x}}[MR_v - c]$

$\leftrightarrow \frac{d\Pi}{dz} = [MR_z - m] + 2\frac{\partial v}{\partial \overline{x}}[MR_v - c]$ (because $z - 2\overline{x}$ is constant)

$\leftrightarrow$ By equations (14), (24), and (24):

$$\frac{d\Pi}{dz} = [\frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}g(z,\overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K} - m] + 2\frac{1+\overline{\theta}}{2}g(z,\overline{x})[\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} - c]$$

$\rightarrow$ Because $\overline{\theta}$ is held constant:

$$\frac{d_2\Pi}{dz^2} = \left([\frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}] + 2\frac{1+\overline{\theta}}{2}[\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} - c]\right)\frac{d}{dz}g(z,\overline{x}(z,v))$$

$\rightarrow \frac{d_2\Pi}{dz^2}(z^*, v^*) = \frac{m\frac{d}{dz}g(z,\overline{x}(z,v))}{g(z,\overline{x}(z,v))}$ (because $\frac{d\Pi}{dz} = 0$ and $\frac{d\Pi}{dv} = 0$ in equilibrium)

$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}(\frac{d}{dz}g(z,\overline{x}(z,v)))$ (because $m > 0$ and $g(z,\overline{x}(z,v)) > 0$)

$\leftrightarrow$ Because $z - 2\overline{x}$ is constant:

$\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}([g(z+1,\overline{x}) - g(z,\overline{x})] + 2[g(z,\overline{x}) - g(z,\overline{x}-1)])$

$\leftrightarrow$ By Lemma A-1 on pg. 34:

$\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}([(\frac{\phi}{1+\phi}g(z,\overline{x}-1) + \frac{1}{1+\phi}g(z,\overline{x})) - g(z,\overline{x})] + 2[g(z,\overline{x}) - g(z,\overline{x}-1)])$

$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}((2 - \frac{\phi}{1+\phi})[g(z,\overline{x}) - g(z,\overline{x}-1)])$

$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}(g(z,\overline{x}) - g(z,\overline{x}-1))$ (because $2 > \frac{\phi}{1+\phi}$)

Note that $g(z,x)$ is the probability mass function of the binomial distribution of x (with a success rate of $\frac{1}{1+\phi}$), which is increasing in x when $x < \frac{\phi}{1+\phi}z$ and decreasing in x when $x > \frac{\phi}{1+\phi}z$. Because $\overline{\theta} = \phi^{z-2x} \rightarrow x = \frac{z-\ln(\overline{\theta})/\ln(\phi)}{2}$, I have that: $x \gtreqless \frac{\phi}{1+\phi}z \leftrightarrow \frac{z-\ln(\overline{\theta})/\ln(\phi)}{2} \gtreqless \frac{\phi}{1+\phi}z \leftrightarrow z \gtreqless \frac{\ln(\overline{\theta})/\ln(\phi)}{2[\frac{1}{2}-\frac{\phi}{1+\phi}]}$

Therefore I have that $\frac{d_2\Pi}{dz^2} > 0$ when $z < \frac{\ln(\overline{\theta})/\ln(\phi)}{2[\frac{1}{2}-\frac{\phi}{1+\phi}]}$ and $\frac{d_2\Pi}{dz^2} < 0$ when $z > \frac{\ln(\overline{\theta})/\ln(\phi)}{2[\frac{1}{2}-\frac{\phi}{1+\phi}]}$. Therefore there is one unique $(z^*, \overline{\theta})$, which means that there is one unique (z^*, v^*). □

This Lemma 6 and Lemma 3 ensure that all firms that do marketing research have the same number of signals, z^*, and the level of advertising, v^*.

Note that these conditions only hold for interior values of z^* and v^*, or equivalently when a firm is targeted advertising. Therefore I need conditions for when I have a boundary condition.

Lemma 7. $\overline{x}(z^*, v^*) \in (\frac{\phi}{1+\phi}z^*, \frac{1}{1+\phi}z^*)$

Proof: Lemma 6 shows $\overline{\theta}$ is unique, thus $z - 2\overline{x}$ is constant. From the proof of Lemma 6, I have $\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}(g(z, \overline{x}) - g(z, \overline{x} - 1))$. By similar reasoning, I have that $\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*, v^*)) = \text{sgn}(\overline{\theta}(z, \overline{x})g(z, \overline{x}) - \overline{\theta}(z, \overline{x}-1)g(z, \overline{x}-1))$ (see Lemma A-3 on pg. 35). Therefore I have that $\text{sgn}(g(z^*, \overline{x}(z^*, v^*)) - g(z^*, \overline{x}(z^*, v^*) - 1)) = \text{sgn}(\overline{\theta}(z, \overline{x})g(z, \overline{x}) - \overline{\theta}(z, \overline{x} - 1)g(z, \overline{x} - 1))$. Note that $\overline{\theta}(z, \overline{x})g(z, x)$ is the probability mass function of the binomial distribution of x (with a success rate of $\frac{\phi}{1+\phi}$), which is increasing in x when $x < \frac{1}{1+\phi}z$ and decreasing in x when $x > \frac{1}{1+\phi}z$. And that $g(z, x)$ is the probability mass function of the binomial distribution of x (with a success rate of $\frac{1}{1+\phi}$), which is increasing in x when $x < \frac{\phi}{1+\phi}z$ and decreasing in x when $x > \frac{\phi}{1+\phi}z$. □

Lemma 7 ensures that any boundary condition only occurs where $z = 0$. This will become a condition on m to ensure a targeted advertising equilibrium once I solve for the targeted advertising price index.

Next I will solve for the price index, using the zero-profit condition. Each entrant j would make the expected marginal profit of π_{M} on each of the $\frac{1}{1+\theta^*(z,v)}v$ consumers that it advertises to who match its type and π_{U} on each of the $\frac{\theta^*(z,v)}{1+\theta^*(z,v)}v$ consumers that it advertises to who do not match its type. Therefore by free entry, I have the zero-profit condition:

$$F + mz = v[\frac{1}{1 + \theta^*(z, v)}\pi_{\text{M}} + \frac{\theta^*(z, v)}{1 + \theta^*(z, v)}\pi_{\text{U}}] \tag{29}$$

Let n_0 be the number of entrants of type 0 and n_1 be the number of entrants of type 1. Solving for n_0 and n_1 equation (29) becomes:

$$n_{TA} \equiv n_0 = n_1 = \frac{\mu}{2(F + mz^* + cv^*)} - \frac{1}{\alpha 2v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]} \tag{30}$$

where $\theta^* \equiv \theta^*(z^*, \overline{x}(z^*, v^*))$, v^* is defined by equation (27), and z^* is defined by equation (28b).

Here $\frac{\mu}{2(F+mz^*+cv^*)}$ is the number of firms that would enter each sub-market if there were no outside option. It is also where entry cost F, marginal marketing research cost m, and marginal advertising cost c effects entry. And $\frac{1}{2\alpha v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}$ is the number of firms discouraged from entering each sub-market due to the outside option.

Note that, unlike (20), this term depends on the amount of marketing research z^* and the amount of advertising v^*, because here firms are using marketing research to decide which consumers to show their ads.

Using (30) to solve for K, I have:

$$K_{TA} = \frac{\alpha\mu}{F + mz^* + cv^*} v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] \tag{31}$$

From this I have:

$$\frac{\partial K_{TA}}{\partial z^*} = 0 \tag{32a}$$

$$\frac{\partial K_{TA}}{\partial v^*} = 0 \tag{32b}$$

I will show in section **??**, that this is because the advertiser is choosing advertising v^* and marketing research z^* to maximize the price index K.

All this assumes interior values of z^* and v^*. Lemma 7 shows that the only possible border solution occurs where the number of signals $z = 0$. Therefore for interior values of z^* and v^*, all I need is for m to be small enough so that the number of signals $z \geq 1$. This condition is given by:

$$m \leq \frac{\frac{1}{2}\frac{1-\phi}{(1+\phi)^2}[4(F + c/2)\frac{e^{b/\mu}}{\frac{1}{1+\phi}e^{b\mu}+\frac{\phi}{1+\phi}} - c]}{1 - 2\frac{1-\phi}{(1+\phi)^2}\frac{e^{b/\mu}}{\frac{1}{1+\phi}e^{b\mu}+\frac{\phi}{1+\phi}}} \tag{33}$$

Note that equation (33) is only the condition for K_{TA} to exist. It does not determine whether an entrant would choose to mass advertise or target advertise. That will be determined by whether targeted advertising covers the fixed cost F or not. I will analyze this in section 4.4.

4.3. Mixed Advertising

In this section, I consider the case some entrants do no marketing research while some entrants do marketing research, or equivalently the case where some entrants mass advertise and some entrants target advertise. By Lemma 3, I have that entrants are indifferent between targeted and mass advertising.

Let n^0_{TA} be the number of type 0 entrants that target advertise, n^0_{MA} be the number of type 0 entrants that mass advertise, n^1_{TA} be the number of type 1 entrants that target advertise, and n^1_{MA} be the number of type 1 entrants that mass advertise. Then by free entry, I have that both the mass advertising zero profit condition, which states $F = \frac{1}{2}[\pi_{\mathrm{M}} + \pi_{\mathrm{U}}]$, and the targeted advertising zero profit condition, which states $F + mz^* = v^*[\frac{1}{1+\theta^*}\pi_{\mathrm{M}} + \frac{\theta^*}{1+\theta^*}\pi_{\mathrm{U}}]$, would be satisfied. This reduces to:

$$n_{MA} = \overline{n}_{MA} + 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA} = 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1} n_{TA} \tag{34}$$

$$\frac{n^1_{MA} - n^0_{MA}}{n^1_{TA} - n^0_{TA}} = -2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1} \tag{35}$$

where $\theta^* \equiv \theta^*(z^*, \overline{x}(z^*, v^*))$, v^* is defined by equation (27), and z^* is defined by equation (28b), $\overline{n}_{MA} \equiv \frac{n^0_{MA}+n^1_{MA}}{2}$, $\overline{n}_{TA} \equiv \frac{n^0_{TA}+n^1_{TA}}{2}$, n_{MA} is the number of entrants in a mass advertising equilibrium, which is given by equation (20), and n_{TA} is the number of entrants in a targeted advertising equilibrium, which is given by equation (30).

Equation (34) bounds $(\overline{n}_{MA}, \overline{n}_{TA})$ to a line between $(n_{MA}, 0)$ and $(0, n_{TA})$. This ensures that advertisers are indifferent between mass advertising and targeted advertising, which results in Lemma 8 (below). Equation (35) assures that the line between (n^0_{MA}, n^0_{TA}) and (n^1_{MA}, n^1_{TA}) has a particular negative slope. This ensures that the price indices from each sub-market are equal, and therefore no entrant is induced to switch product type. Interestingly, this allows one sub-market to have more mass advertisers while the other sub-market to have more targeted advertisers. There is even one equilibrium where all entrants in one sub-market mass advertise and all entrants in the other market target advertise. For an examination of equations (34) and (35), see Appendix C.

Lemma 8. *If entrants are indifferent between targeted and mass advertising, then:*

$$K_{mixed} = K_{MA} = K_{TA}$$

Proof: Part 1: Proof of $K_{mixed} = K_{MA}$

From (34) I have: $n_{MA} = \overline{n}_{MA} + 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{+\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA} \leftrightarrow K_{MA} = 1 + \alpha n_{MA}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]$
$= 1 + \alpha \overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] + 2\alpha \overline{n}_{TA} v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] = K_{mixed}$

Part 2: Proof of $K_{mixed} = K_{TA}$

From (34) I have: $2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1} n_{TA} = \overline{n}_{MA} + 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA}$
$\leftrightarrow n_{TA} = \frac{e^{b/\mu}+1}{2v^*[\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}]}\overline{n}_{MA} + \overline{n}_{TA} \leftrightarrow K_{TA} = 1 + 2\alpha n_{TA} v^*(\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}})$
$\leftrightarrow K_{TA} = 1 + \alpha \overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] + 2\alpha \overline{n}_{TA} v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] = K_{mixed}$ □

Lemmas 6 and 8 gives us that (z^*, v^*) must be unique.

4.4. Equilibrium Uniqueness

Putting the results from sections 4.1, 4.2, and 4.3 together, I have established a unique equilibrium:

Proposition 1. *Equilibrium is determined by one unique* K_{TA} *defined by equation* (31) *and one unique* K_{MA} *defined by equation* (21).

(a) *When* $K_{MA} > K_{TA}$*, there exists one uniqe equilibrium, where:*

- *There are* n_{MA} *entrants of each type, where* n_{MA} *is uniquely defined by equation* (20)*.*
- *Each entrant chooses its price* $p = \mu$*, does no marketing research* ($z = 0$)*, and mass advertises.*

(b) *When* $K_{TA} > K_{MA}$*, there exists one uniqe equilibrium, where:*

- *There are* n_{TA} *entrants of each type, where* n_{TA} *is uniquely defined by* (30)*.*
- *Each entrant chooses its price* $p = \mu$*, does* $z = z*$ *marketing research and* $v = v^*$ *advertising, where* z^* *is uniquely defined by equation* (28b) *and* v^* *is uniquely defined by equation* (27)*.*

(c) *When* $K_{MA} = K_{TA}$*, there exist multiple potential equilibria. The necessary and sufficient conditions for an equilibrium are:*

- *The total number of entrants of each type solves equations* (34) *and* (35)*.*
- *Each entrant chooses its price* $p = \mu$*, chooses to either target advertise or mass advertise, and is indifferent between targeted and mass advertising.*
- *Those entrants that target advertise choose* $z = z*$ *and* $v = v^*$*, where* z^* *is uniquely defined by equation* (28b) *and* v^* *is uniquely defined by equation* (27)*.*

Proof: I have established that there are three potential equilibria: (1) All entrants mass advertise, (2) All entrants target advertise, and (3) entrants are indifferent between targeted and mass advertising. And I have established that K_{TA} is uniquely defined by equation (31) and K_{MA} is uniquely defined by equation (21).

If all entrants mass advertise, then $K = K_{MA}$ and it must be the case that:

$v^*[\frac{1}{1+\theta^*}\pi_M + \frac{\theta^*}{1+\theta^*}\pi_U] - F - mz^* \leq 0$ (or else entrants would target advertise)

$\leftrightarrow F + mz^* + cv^* \geq \alpha\mu v^* \frac{\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}}{K_{MA}}$

$\leftrightarrow K_{MA} \geq \frac{\alpha\mu}{F+mz^*+cv^*}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] \leftrightarrow K_{MA} \geq K_{TA}$

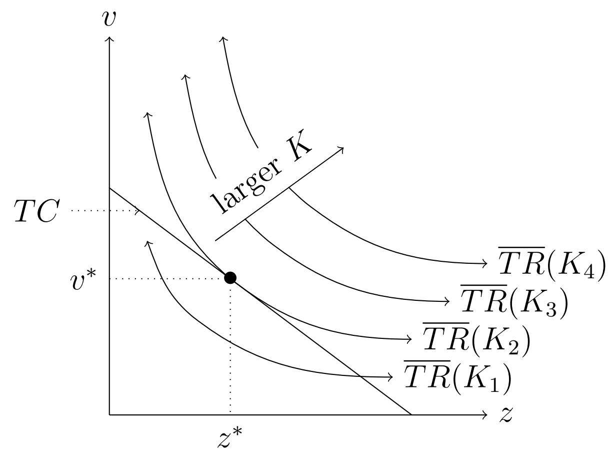
If all entrants target advertise then $K = K_{TA}$ and it must be the case that:

$\frac{1}{2}\pi_{\text{M}} + \frac{1}{2}\pi_{\text{U}} - F \leq 0 \leftrightarrow \frac{1}{2}[\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K_{TA}} - c] + \frac{1}{2}[\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K_{TA}} - c] - F \leq 0$

$\leftrightarrow F + c \geq \alpha\mu\frac{\frac{1}{2}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]}{K_{TA}} \leftrightarrow K_{TA} \geq \frac{\alpha\mu}{2(F+c)}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] \leftrightarrow K_{TA} \geq K_{MA}$

Lemma 8 shows that if entrants are indifferent between targeted and mass advertising, $K_{mixed} = K_{MA} = K_{TA}$. □

Figure 2: The Dual Problem



5. Equilibrium Analysis

In this section, I explore the equilibrium found by Proposition 1 with a particular emphasis on the Targeted Advertising Equilibrium.

Until now, we have looked at the problem where advertisers choose marketing research z and advertising v to maximize their profit given the price index K. Now we will be looking at an alternative way of looking at the advertiser's problem, which will yield several additional comparative statics. Note that in equilibrium I must have the advertiser's total revenue $TR(z, v, K) = \alpha\mu v \frac{\frac{1}{1+\theta^*(z,v)} e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)} e^{\frac{R-\mu}{\mu}}}{K}$ equal to its total cost $TC(z, v) = F + mz + cv$, or equivalently the zero-profit condition given by equation (29). Further note that $\frac{\partial}{\partial K} TR(z, v, K) < 0$, which is equivalent to the iso-revenue curve shifting upward and rightward with an increase in K, as depicted in Figure 2.

Therefore in equilibrium, marketing research z, advertising v, and price index K solve the dual problem:[14]

$$\max_{z,v,K} K \tag{36}$$

$$\text{subject to } F + mz = v\left[\frac{1}{1+\theta^*(z,v)}\pi_{\mathrm{M}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)}\pi_{\mathrm{U}}\right] \tag{37}$$

This is consistent with Currier (2002) who made a similar argument to show that the choices of production inputs maximize K subject to the zero-profit condition in similar monopolistically competitive models. This yields a number of interesting comparative statics that show how similar production substitution and information substitution are.

By the envelop theorem on the dual problem given by equations (36), I have the effect of changing the marginal marketing research cost m, the marginal advertising cost c, and the

[14] This also works for the Mass or Mixed advertising equilibria.

fixed cost F on the price index K:

$$\mathcal{E}_{K,m} = -\frac{mz^*}{F + mz^* + cv^*} < 0 \tag{38a}$$
$$\mathcal{E}_{K,c} = -\frac{cv^*}{F + mz^* + cv^*} < 0 \tag{38b}$$
$$\mathcal{E}_{K,F} = -\frac{F}{F + mz^* + cv^*} < 0 \tag{38c}$$

where $\mathcal{E}_{K,\rho}$ is the elasticity of the price index K with respect to changing $\rho \in \{m, c, F\}$.

Therefore the elasticity of price index K with respect to changing any input price (market research, advertising, or fixed cost) is equal to negative of the proportion of the total cost from that input. Das (1981) and Currier (2002) found a similar result for the affect for production inputs in monopolistic competition.

I use this to solve for how z and $\overline{x}$ change with marketing research cost:

$$\mathcal{E}_{z^*,m} = \frac{\frac{1+\overline{\theta}e^{-b/\mu}}{2}\mathcal{E}_{\pi_M,m} + \frac{1+\overline{\theta}}{3-\overline{\theta}}(1 - \mathcal{E}_{\pi_M,m})}{z^* \ln\phi(\frac{1}{2} - \frac{\phi}{1+\phi})} < 0 \tag{39a}$$
$$\mathcal{E}_{\overline{x},m} = \frac{\frac{\phi}{1+\phi}\frac{1+\overline{\theta}e^{-b/\mu}}{2}\mathcal{E}_{\pi_M,m} + \frac{1}{2}\frac{1+\overline{\theta}}{3-\overline{\theta}}(1 - \mathcal{E}_{\pi_M,m})}{\overline{x} \ln\phi(\frac{1}{2} - \frac{\phi}{1+\phi})} < 0 \tag{39b}$$

where the elasticity of π_M with respect to marketing research cost $\mathcal{E}_{\pi_M,m} = -\mathcal{E}_{K,m}\frac{\pi_M+c}{\pi_M} \in (0,1)$, and $\overline{\theta} \in (0,3)$ (by Lemma A-9 on page 35). Notice that when m changes, $\overline{x}$ changes for two reasons: (1) z^* is changing, and (2) v^* is changing. A percentage increase in m would cause an $\mathcal{E}_{z^*,m}$ change in $z*$, which would cause a $\mathcal{E}_{\overline{x},z}\mathcal{E}_{z^*,m}$ change in $\overline{x}$, where $\mathcal{E}_{\overline{x},z}$ is the elasticity of $\overline{x}$ with respect to z (holding v constant). Also a percentage increase in m would cause an $\mathcal{E}_{v^*,m}$ change in $v*$, which would cause a $\mathcal{E}_{\overline{x},v}\mathcal{E}_{v^*,m}$ change in $\overline{x}$, where $\mathcal{E}_{\overline{x},v}$ is the elasticity of $\overline{x}$ with respect to v (holding z constant). Therefore $\mathcal{E}_{\overline{x},m} = \mathcal{E}_{\overline{x},z}\mathcal{E}_{z^*,m} + \mathcal{E}_{\overline{x},v}\mathcal{E}_{v^*,m}$, which becomes:

$$\begin{aligned}\mathcal{E}_{v^*,m} &= \mathcal{E}_{v,\overline{x}}(\mathcal{E}_{\overline{x},m} - \mathcal{E}_{\overline{x},z}\mathcal{E}_{z^*,m}) \\ &= \frac{1+\overline{\theta}}{2}g(z,\overline{x})\frac{[\frac{\phi}{1+\phi} - \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})}]\frac{1+\overline{\theta}e^{-b/\mu}}{2}\mathcal{E}_{\pi_M,m} + [\frac{1}{2} - \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})}]\frac{1+\overline{\theta}}{3-\overline{\theta}}(1 - \mathcal{E}_{\pi_M,m})}{v^* \ln\phi(\frac{1}{2} - \frac{\phi}{1+\phi})}\end{aligned} \tag{40}$$

where $\frac{d\overline{x}}{dm} < 0$ is given by equation (39b), $\frac{\partial\overline{x}}{\partial z} > 0$ is given by equation (16), and $\frac{dz^*}{dm} < 0$ is given by equation (39a).

Equation (39a) shows that z has the classic downward sloping demand curve, despite the fact that increasing the cost of marketing research indirectly increases the value of advertising by encouraging less competition. Equation (40) shows that advertising can be either a substitute or a complement to marketing research: If an advertiser is sending ads to a small targeted group of consumers (which needs to be sufficiently less than half of the consumers), then increasing the cost of marketing research could induce the advertiser to substitute more ads for less marketing research. However, if the advertiser is sending ads to a less targeted group of consumers, then increasing the cost of marketing research would induce the advertiser to not only do less marketing research but do less advertising as advertising would be less effective.

Equations (39a) and (40) also show that: (1) for small m: increasing m increases the amount an advertiser spends on advertising v and marketing research z, and (2) for large m: increasing m decreases the amount an advertiser spends on advertising v and marketing research z. This is important because the number of advertisers can be rewritten as:

$$\text{Targeted and Mass Advertising Equilibria}: \quad n = \frac{\mu}{2TC(z^*, v^*)}[1 - \frac{1}{K}] \tag{41}$$

$$\text{Mixed Advertising Equilibria}: \quad TC(1,1)\overline{n}_{MA} + TC(z^*, v^*)\overline{n}_{TA} = \frac{\mu}{2}[1 - \frac{1}{K}] \tag{42}$$

Therefore I have that (1) for small m: increasing m decreases n (because TC is increasing) and (2) for large n: increasing m increases n (because TC is decreasing. This is important because the number n of firms affects the welfare analysis of consumers, advertiser platforms, and marketing research firms.

6. Conclusion

In conclusion, I have found several similarities between the advertising-marketing trade-off and the trade-off between production inputs. The Marginal Rate of Information substitution is downward sloping and convex (where firms optimize), similar to the Marginal Rate of Technical Substitution. As a result, the cost minimization and profit maximization problems are similar. I found similar properties of the Price Index for a monopolistically competitive firm deciding how much advertising and marketing research to use and a monopolistically competitive firm deciding how much inputs to use. This includes the discovery that both the quantities of advertising and marketing research maximize the price index, which is similar to a discovery by Currier (2002) for production inputs.

In future papers, could this be extended to an advertising-production input trade-off or a marketing-production input trade-off? Possibly. Although, changing the quantities of advertising and marketing research shift the demand for a product, and changing the quantities of a production input would move us along the demand for a product (as it only changes quantity supplied). More likely, this could be extended to an examination of the substitution between all demand shifters, which may prove useful as some demand shifters (like advertising and marketing research) are controllable by the firm and other demand shifters (like prices of other goods) are controllable by other entities.

Also I done some preliminary work on the welfare implications of changing m. In particular, I found that in industries where the marketing research is controlled by a single ad platform (like Google), privacy regulations would always hurt consumers. The ad platform's incentive for higher profits would impose stricter privacy controls than a government regulator trying to maximize consumer welfare. This is particularly important because many ad platforms act as a monopoly bottleneck to consumers.

Also I have shown that there is one unique Equilibrium (either a mass advertising Equilibrium or a Targeted advertising Equilibrium), with the exception of the razor-edge case where marketing research costs are at a threshold value (which yields a continuum of possible mixed advertising Equilibra). This is important because multiple eqilibria would complicate comparative static analysis and policy analysis.

Also I have created a very flexible framework that could be used to look at different aspects of targeted advertising. I have created a monopolistically competitive logit model where it is easy to look at imperfect targeted advertising. In a previous version of this paper, I used a simpler version of the model, where advertisers get one noisy signal on consumers (if they choose to target advertise), to look at the effects of ad taxes, endogenous advertising prices, ad retention, and shared information on a noisy targeted advertising market. This paper is a narrow look at one of the extensions in that previous version of this paper.

Also I have developed the binomial distribution as a way of aggregating multiple signals. Most of the literature on discrete economics focuses on a zero-one decision (for example: Anderson et al., 1992). In this paper, I explored how we can use the binomial distribution to aggregate many zero-one decisions to make an integer decision. While much of the analysis using the binomial distribution was similar, there were some differences. For example, instead of first-order equations, I used first-difference equations.

Also this paper emphasizes the need for empirical and theoretical developments of the ad annoyance function. This is critical in understanding how privacy regulations affect consumer welfare. Are consumers more annoyed by the first ad or the second? This empirical question needs to be answered for us to understand whether we want to have an intermediate-level signal accuracy, or an extreme. And it needs to be answered for us to understand whether privacy regulations should limit the information firms can gather or make it more expensive for firms to gather additional information.

Ad annoyance is a possible explanation for why we value privacy, why we don't want firms to know too much information about us. While we may never be able to test empirically why people value privacy, we shouldn't claim that it has to be purely an intermediate good or purely from the fear of the criminal use of our information.

Also there is a need to study different marketing research technologies than the binomial marketing research technology that I have discussed here. For example, if firms were to stop gathering signals from consumers after a fixed number of unmatched signals and advertise to consumers after a fixed number of matched signals, then the negative binomial distribution would be more appropriate than the binomial distribution. This would better fit situations where firms go through a search process for consumers than gather a fixed number of signals from every consumer. Also continuous signals distributed normally would aggregate to a normal distribution, which might prove easier to handle. Also additional spending on marketing research does not necessarily yield additional information on a consumer through additional signals. It could be that additional marketing research yields better quality signals. Under this interpretation any distribution could be used as long as additional spending reduces the error of a signal.

Also there is potential in adapting this model to other market structures, like monopoly, perfect competition, and oligopoly. Although there has been other research in targeted advertising in other market structures, the flexibility of this model may prove helpful in further studies of other markets.

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Appendix A. Proofs of Equations

The following are the algebraic proofs for the equations in the text as needed.

Proof of equation (4)*:* By Baye's Theorem: $\Pr(\tau_i = t_j | x_{ij} = k) = \frac{\frac{1}{2}g(z_j,k)}{\frac{1+\theta_{ij}}{2}g(z_j,k)} = \frac{1}{1+\theta_{ij}}$

$\leftrightarrow \Pr(\tau_i = 1 - t_j|\theta_{ij}) = 1 - \Pr(\tau_i = t_j|\theta_{ij}) = \frac{\theta_{ij}}{1+\theta_{ij}}$ □

Proof of equation (8)*:* From equation (7), I have that the first-order condition $\frac{\partial}{\partial p_j}\pi(p_j, t_j, \theta_{ij}) = 0$ is equivalent to $p_j = \mu$:

$\frac{\partial}{\partial p_j}\pi(p_j, t_j, \theta_{ij}) = 0 \leftrightarrow \alpha[\frac{\frac{1}{1+\theta_{ij}}e^{\frac{R+b-p_j}{\mu}}}{K_{t_j}} + \frac{\frac{\theta_{ij}}{1+\theta_{ij}}e^{\frac{R-p_j}{\mu}}}{K_{1-t_j}}] - \frac{p}{\mu} * \alpha[\frac{\frac{1}{1+\theta_{ij}}e^{\frac{R+b-p_j}{\mu}}}{K_{t_j}} + \frac{\frac{\theta_{ij}}{1+\theta_{ij}}e^{\frac{R-p_j}{\mu}}}{K_{1-t_j}}] = 0$

$\leftrightarrow 1 - \frac{p_j}{\mu} = 0 \leftrightarrow p_j = \mu$ □

Proof of equation (16)*:* $0 = \frac{dv}{d\overline{x}}d\overline{x} + \frac{dv}{dz}dz \leftrightarrow \frac{d\overline{x}}{dz} = -\frac{dv/dz}{dv/d\overline{x}}$

$\leftrightarrow \frac{d\overline{x}}{dz} = -\frac{-\frac{1}{2}\frac{\overline{\theta}+\phi}{1+\phi}g(z,x)}{\frac{1+\overline{\theta}}{2}g(z,\overline{x})}$ $\leftrightarrow$ Equation (16) (by equations (14) and (15)) □

Proof of equation (17)*:* $\frac{1}{2}G_1(z,\overline{x}) = \frac{1}{2}[G_1(z,\overline{x}) + G_\phi(z,\overline{x})]\frac{G_1(z,\overline{x})}{G_1(z,\overline{x})+G_\phi(z,\overline{x})}$

$= v\frac{G_1(z,\overline{x})}{G_1(z,\overline{x})+G_\phi(z,\overline{x})} = v\frac{1}{1+\theta^*(z,v)}$ (by equation (13))

$\frac{1}{2}G_\phi(z,\overline{x}) = \frac{1}{2}[G_1(z,\overline{x}) + G_\phi(z,\overline{x})]\frac{G_\phi(z,\overline{x})}{G_1(z,\overline{x})+G_\phi(z,\overline{x})}$

$= v\frac{G_\phi(z,\overline{x})}{G_1(z,\overline{x})+G_\phi(z,\overline{x})} = v\frac{\theta^*(z,v)}{1+\theta^*(z,v)}$ (by equation (13)) □

Proof of equation (18): $\frac{\mu}{2(F+c)} - \frac{1}{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]} > 1 \leftrightarrow \frac{\mu}{2(F+c)} > \frac{1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}$
$\leftrightarrow \frac{2(F+c)}{\mu} < \frac{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}{1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]} \leftrightarrow F+c < \frac{\mu}{2}\frac{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}{1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}$
$\leftrightarrow \alpha\mu[\frac{\frac{1}{2}e^{\frac{R+b-\mu}{\mu}}+\frac{1}{2}e^{\frac{R-\mu}{\mu}}}{1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}] - F - c > 0.$

Let's assume that one firm enters as type $1-t$, then it is sufficient to show that it is profitable for another firm to enter as type t to prove the lemma. Lets call the firm that potentially enters as type t firm t and the firm that enters as type $1-t$ firm $1-t$. Let z_{1-t} and v_{1-t} be firm $1-t$'s choice of z and v.

If firm t were to send ads to all consumers, then
$K_t = 1+\alpha[e^{\frac{R+b-\mu}{\mu}} + \frac{2\theta^*(z_{1-t},v_{1-t})}{1+\theta^*(z_{1-t},v_{1-t})}v_{1-t}e^{\frac{R-\mu}{\mu}}] < 1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]$, and
$K_{1-t} = 1+\alpha[e^{\frac{R-\mu}{\mu}} + \frac{2}{1+\theta^*(z_{1-t},v_{1-t})}v_{1-t}e^{\frac{R+b-\mu}{\mu}}] < 1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]$.
Then $\Pi(t,0,0) = \alpha\mu[\frac{\frac{1}{2}e^{\frac{R+b-\mu}{\mu}}}{K_t} + \frac{\frac{1}{2}e^{\frac{R-\mu}{\mu}}}{K_{1-t}}] - F - c > \alpha\mu[\frac{\frac{1}{2}e^{\frac{R+b-\mu}{\mu}}+\frac{1}{2}e^{\frac{R-\mu}{\mu}}}{1+\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}] - F - c$, which I have shown is greater than 0. Therefore firm t would enter the martet. And by symmetry firm $1-t$ would enter the martet. □

Proof of equation (20): By Lemma 3, $K_0 = K_1$
$\leftrightarrow 1+\alpha(n_0 e^{\frac{R+b-\mu}{\mu}} + n_1 e^{\frac{R-\mu}{\mu}}) = 1+\alpha(n_0 e^{\frac{R-\mu}{\mu}} + n_1 e^{\frac{R+b-\mu}{\mu}}) \leftrightarrow n_0 = n_1 \equiv n_{MA}$.
By free entry, I have the zero profit condition
$\leftrightarrow F = \frac{1}{2}\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}}{K} - c \leftrightarrow F+c = \frac{1}{2}\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}}{1+\alpha n_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}$
$\leftrightarrow \frac{2(F+c)}{\mu} = \frac{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}{1+\alpha n_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]} \leftrightarrow \frac{\mu}{2(F+c)} = \frac{1}{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]} + n_{MA} \leftrightarrow (20)$ □

Proof of equation (21): By equation (20), I have $K_{MA} = 1+\alpha n_{MA}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]$
$= 1+\alpha\left(\frac{\mu}{2(F+c)} - \frac{1}{e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}}\right)[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] \leftrightarrow (21)$ □

Proof of equation (24):
$MR_z = -\frac{\alpha\mu}{2}g(z,\overline{x}(z,v))\frac{\frac{\phi}{1+\phi}e^{\frac{R+b-\mu}{\mu}}+\frac{1}{1+\phi}\overline{\theta}e^{\frac{R-\mu}{\mu}}}{K} + \frac{1}{2}[\frac{\phi}{1+\phi}+\frac{1}{1+\phi}\overline{\theta}]g(z,\overline{x}(v,z))MR_v$
$= -\frac{\alpha\mu}{2}g(z,\overline{x}(z,v))\frac{\frac{\phi}{1+\phi}e^{\frac{R+b-\mu}{\mu}}+\frac{1}{1+\phi}\overline{\theta}e^{\frac{R-\mu}{\mu}}}{K} + \frac{1}{2}g(z,\overline{x}(z,v))[\frac{\phi}{1+\phi}+\frac{1}{1+\phi}\overline{\theta}]*\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}}+\frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K}$
$= -\frac{\alpha\mu}{2}g(z,\overline{x}(z,v))\frac{\frac{\phi}{1+\phi}e^{\frac{R+b-\mu}{\mu}}+\frac{1}{1+\phi}\overline{\theta}e^{\frac{R-\mu}{\mu}}}{K} + \frac{\alpha\mu}{2}\frac{\overline{\theta}+\phi}{1+\phi}g(z,\overline{x}(z,v))\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}}+\frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K}$

$= \frac{\alpha\mu}{2} g(z, \overline{x}(z,v))[(\frac{\overline{\theta}+\phi}{1+\phi}\frac{1}{1+\overline{\theta}} - \frac{\phi}{1+\phi}) * \frac{e^{\frac{R+b-\mu}{\mu}}}{K} + (\frac{\overline{\theta}+\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}} - \frac{\overline{\theta}}{1+\phi}) * \frac{e^{\frac{R-\mu}{\mu}}}{K}]$

$= \frac{\alpha\mu}{2} g(z, \overline{x}(z,v))[\frac{(\overline{\theta}+\phi)-\phi(1+\overline{\theta})}{(1+\phi)(1+\overline{\theta})}\frac{e^{\frac{R+b-\mu}{\mu}}}{K} + \frac{\overline{\theta}(\overline{\theta}+\phi)-\overline{\theta}(1+\overline{\theta})}{(1+\phi)(1+\overline{\theta})}\frac{e^{\frac{R-\mu}{\mu}}}{K}] \leftrightarrow$ Equation (24) □

Proof of equation (25)*:* $MRIS_{zv} = \frac{MR_z}{MR_v} = \frac{\frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}} g(z,\overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}}{\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K}} \leftrightarrow (25)$ □

Proof of equation (27)*:* $MR_v = c \leftrightarrow \alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} = c \leftrightarrow \pi(\overline{\theta}) = 0$ and $\overline{\theta} = \frac{\pi_{\mathrm{M}}}{-\pi_{\mathrm{U}}}$ □

Proof of equation (28a)*:* $m = \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}} g(z, \overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}$

$\leftrightarrow m = \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\pi_{\mathrm{M}}}{\pi_{\mathrm{M}}-\pi_{\mathrm{U}}} g(z, \overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}$ (by equation (27))

$\leftrightarrow m = \frac{1}{2}\frac{1-\phi}{1+\phi}\frac{\pi_{\mathrm{M}}}{\pi_{\mathrm{M}}-\pi_{\mathrm{U}}} g(z, \overline{x}(z,v))(\pi_{\mathrm{M}} - \pi_{\mathrm{U}}) \leftrightarrow$ (28b) □

Proof of equation (30)*:* By Lemma 3, $K_0 = K_1 = 1 + 2\alpha v^*(\frac{1}{1+\theta^*}n_0 e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}n_1 e^{\frac{R-\mu}{\mu}})$
$= 1 + 2\alpha v^*(\frac{1}{1+\theta^*}n_1 e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}n_0 e^{\frac{R-\mu}{\mu}}) \leftrightarrow n_0 = n_1 \equiv n_{TA}$.
By free entry, I have the zero profit condition:

$F + mz^* = v^*[\frac{1}{1+\theta^*}\pi_{\mathrm{M}} + \frac{\theta^*}{1+\theta^*}\pi_{\mathrm{U}}] = v^*[\frac{1}{1+\theta^*}(\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K} - c) + \frac{\theta^*}{1+\theta^*}(\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K} - c)]$

$\leftrightarrow F + mz^* + cv^* = \alpha\mu v^*\frac{\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}}{K} = \alpha\mu v^*\frac{\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}}{1+2\alpha n_{TA}v^*(\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}})}$

$\leftrightarrow \frac{\mu}{2(F+mz^*+cv^*)} = n_{TA} + \frac{1}{2\alpha v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{1}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]} \leftrightarrow (30)$ □

Proof of equation (31)*:*
By equation (30), I have: $K_{TA} = 1 + 2\alpha n_{TA}v^*(\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}})$
$= 1 + 2\alpha(\frac{\mu}{2(F+mz^*+cv^*)} - \frac{1}{2\alpha v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]})v^*(\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}) \leftrightarrow (31)$ □

Proof of equations (32a) *and* (32b)*:*
Let $K(x, z) \equiv \frac{\alpha\mu}{F+mz+cv}v[\frac{1}{1+\theta^*(z,v)}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)}e^{\frac{R-\mu}{\mu}}]$.
By equation (24):

$\frac{\partial}{\partial z}K(x, z) = \alpha\mu\frac{\frac{1}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}(z,x)}{1+\overline{\theta}(z,x)}g(z,\overline{x}(z,v))(e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}})*(F+mz+cv) - v[\frac{1}{1+\theta^*(z,v)}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)}e^{\frac{R-\mu}{\mu}}]*m}{(F+mz+cv)^2}$

$= \frac{K(z,v)}{F+mz+cv}[MR_z - m] \rightarrow$ By the first-order condition of z: (32a) and by similar reasoning (32b). □

Proof of equation (33)*:* By Parts:

Part 1: Characteristic of an Interior Solution

K_{TA} exists if and only if (v^*, z^*) is an interior solution. From Lemma 7, I have that the solution for the first-order condition for v, which is given by equation (27), is always an interior solution. Therefore K_{TA} exists if and only if z^* is an interior solution for z, or equivalently when equation (28b) is satisfied for at least one $z^* \geq 1$.

Part 2: At least one threshold value of m exists

Let $K(z,v) \equiv \frac{\alpha\mu}{F+mz+cv} v[\frac{1}{1+\theta^*(z,v)} e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*(z,v)}{1+\theta^*(z,v)} e^{\frac{R-\mu}{\mu}}]$. By equations (32a) and (32b), I have that $\frac{\partial}{\partial m} K(z > 0, v) < 0$, so $\frac{\partial}{\partial m} \pi_M(K = K(z > 0, v)) > 0$.[a] By l'Hôpital's Rule: $\lim_{m\to\infty} \frac{\pi_M}{m} = \lim_{m\to\infty} \frac{d\pi_M}{dm} = 0$. Therefore $m > \frac{1}{2}(\frac{1-\phi}{1+\phi}) g(z, \overline{x}(z,v)) \pi_{\mathrm{M}}$ for all (z,v), when $m \to \infty$. Also when marketing research is free (i.e. $m = 0$), I have that each entrant will buy as many free signals as possible (i.e. $z = \infty$), or equivalently $m < \frac{1}{2}(\frac{1-\phi}{1+\phi}) g(z, \overline{x}(z,v)) \pi_{\mathrm{M}}$ for all (z,v).

Therefore by the intermediate value theorem, there exists at least one threshold value of m, which is the border between values of m where K_{TA} exists and values of m where K_{TA} does not exist.

Part 3: Characteristic of a threshold value of m

From Lemma 7, I have that any such threshold value of m would be where $z = 1$ and $\overline{x} = 0$ (i.e. $v = \frac{1}{2}$). For this value of m, $K_{TA} = K(z = 1, v = \frac{1}{2})$
$= \frac{\alpha\mu}{F+m+c/2} \frac{1}{2}[\frac{1}{1+\phi} e^{\frac{R+b-\mu}{\mu}} + \frac{\phi}{1+\phi} e^{\frac{R-\mu}{\mu}}]$, because all firms would set $z = 1$ and $v = \frac{1}{2}$. Also by the first order condition of z, I have: $m = \frac{1}{2}(\frac{1-\phi}{1+\phi}) g(z = 1, \overline{x} = 0)\pi_{\mathrm{M}} = \frac{1}{2}(\frac{1-\phi}{1+\phi}) \frac{1}{1+\phi} \pi_{\mathrm{M}}$
$= \frac{1}{2}(\frac{1-\phi}{1+\phi}) \frac{1}{1+\phi}[\alpha\mu \frac{e^{\frac{R+b-\mu}{\mu}}}{K(z=1, v=\frac{1}{2})} - c] = \frac{1}{2}(\frac{1-\phi}{1+\phi}) \frac{1}{1+\phi}[\alpha\mu \frac{e^{\frac{R+b-\mu}{\mu}}}{\frac{\alpha\mu}{F+m+c/2}\frac{1}{2}[\frac{1}{1+\phi} e^{\frac{R+b-\mu}{\mu}} + \frac{\phi}{1+\phi} e^{\frac{R-\mu}{\mu}}]} - c]$

$$\leftrightarrow m = \frac{\frac{1}{2}\frac{1-\phi}{(1+\phi)^2}[4(F+c/2)\frac{e^{b/\mu}}{\frac{1}{1+\phi}e^{b\mu} + \frac{\phi}{1+\phi}} - c]}{1 - 2\frac{1-\phi}{(1+\phi)^2}\frac{e^{b/\mu}}{\frac{1}{1+\phi}e^{b\mu} + \frac{\phi}{1+\phi}}}.$$

Because this is unique, there is one threshold value. When $m = 0$, I have shown that K_{TA} exists, and when $m \to \infty$, I have shown that K_{TA} does not exist. Therefore I have that K_{TA} exists if and only if equation (33) is satisfied. □

[a] I will examine this result more in section ??.

Proof of equations (34) *and* (35)*:* Part 1: Proof of (34)

I have that $K_0 = 1 + \alpha[n^0_{MA} e^{\frac{R+b-\mu}{\mu}} + n^1_{MA} e^{\frac{R-\mu}{\mu}}] + 2\alpha v^*[\frac{1}{1+\theta^*} n^0_{TA} e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*} n^1_{TA} e^{\frac{R-\mu}{\mu}}]$ and that $K_1 = 1 + \alpha[n^1_{MA} e^{\frac{R+b-\mu}{\mu}} + n^0_{MA} e^{\frac{R-\mu}{\mu}}] + 2\alpha v^*[\frac{1}{1+\theta^*} n^1_{TA} e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*} n^0_{TA} e^{\frac{R-\mu}{\mu}}]$.
From Lemma 3, I have that:

$K = K_0 = K_1 = \frac{K_0+K_1}{2} = 1 + \alpha\overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}] + 2\alpha\overline{n}_{TA}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]$.

The free entry condition of the mass-advertising entrants is:

$F = \frac{1}{2}[\pi_{\text{M}} + \pi_{\text{U}}] \leftrightarrow F + c = \alpha\mu\frac{\frac{1}{2}(e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}}{K})$

$\leftrightarrow F + c = \alpha\mu\frac{\frac{1}{2}(e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}})}{1+\alpha\overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]+2\alpha\overline{n}_{TA}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}$

$\leftrightarrow \frac{\mu}{2(F+c)} = \frac{1+\alpha\overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]+2\alpha\overline{n}_{TA}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]}$

$\leftrightarrow \frac{\mu}{2(F+c)} = \frac{1}{\alpha[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]} + \overline{n}_{MA} + 2v^*\frac{\frac{1}{1+\theta^*}e^{b/\mu}+\frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA}$

$\leftrightarrow \overline{n}_{MA} + 2v^*\frac{\frac{1}{1+\theta^*}e^{b/\mu}+\frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA} = n_{MA}$, where n_{MA} is defined in equation (20).

The free entry condition of the targeted-advertising entrants is:

$F + mz^* = v^*[\frac{1}{1+\theta^*}\pi_{\text{M}} + \frac{\theta^*}{1+\theta^*}\pi_{\text{U}}] \leftrightarrow F + mz^* + cv^* = \alpha\mu v^*\frac{\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}}{K}$

$\leftrightarrow F + mz^* + cv^* = \alpha\mu v^*\frac{\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}}{1+\alpha\overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]+2\alpha\overline{n}_{TA}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}$

$\leftrightarrow \frac{\mu}{2(F+mz^*+cv^*)} = \frac{1+\alpha\overline{n}_{MA}[e^{\frac{R+b-\mu}{\mu}}+e^{\frac{R-\mu}{\mu}}]+2\alpha\overline{n}_{TA}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}{2\alpha v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]}$

$\leftrightarrow \frac{\mu}{2(F+mz^*+cv^*)} = \frac{1}{2\alpha v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}}+\frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]} + \frac{e^{b/\mu}+1}{2v^*[\frac{1}{1+\theta^*}e^{b/\mu}+\frac{\theta^*}{1+\theta^*}]}\overline{n}_{MA} + \overline{n}_{TA}$

$\leftrightarrow \overline{n}_{MA} + 2v^*\frac{\frac{1}{1+\theta^*}e^{b/\mu}+\frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\overline{n}_{TA} = 2v^*\frac{\frac{1}{1+\theta^*}e^{b/\mu}+\frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}n_{TA}$, where n_{TA} is defined in equation (30).

Part 2: Proof of (35)

$K_0 = K_1$

$\leftrightarrow 1 + \alpha[n^0_{MA}e^{\frac{R+b-\mu}{\mu}} + n^1_{MA}e^{\frac{R-\mu}{\mu}}] + 2\alpha v^*[\frac{1}{1+\theta^*}n^0_{TA}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}n^1_{TA}e^{\frac{R-\mu}{\mu}}]$

$= 1 + \alpha[n^1_{MA}e^{\frac{R+b-\mu}{\mu}} + n^0_{MA}e^{\frac{R-\mu}{\mu}}] + 2\alpha v^*[\frac{1}{1+\theta^*}n^1_{TA}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}n^0_{TA}e^{\frac{R-\mu}{\mu}}]$

$\leftrightarrow 0 = (n^1_{MA}-n^0_{MA})[e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}] + (n^1_{TA}-n^0_{TA})2v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} - \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] \leftrightarrow (35)$ □

Proof of equations (38a) *to* (38c)*:* By the envelop theorem on the dual problem given by equations (36) and (29), I have that $\frac{dK_{TA}}{dm} = \frac{\partial K_{TA}}{\partial m}$.
Equation (31) $\rightarrow \frac{dK_{TA}}{dm} = -z^*\frac{\alpha\mu}{(F+mz^*+cv^*)^2}v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}] \leftrightarrow$ Equation (38a) and by symmetry Equations (38b) and (38c). □

Proof of equations (39a) *and* (39b)*:* I have that $\mathcal{E}_{\pi_M,m} = \mathcal{E}_{\pi_M,K} * \mathcal{E}_{K,m} = -\alpha\mu\frac{e^{\frac{R+b-\mu}{\mu}}}{K^2}\frac{K}{\pi_M} * \mathcal{E}_{K,m} = -\mathcal{E}_{K,m}\frac{\pi_M+c}{\pi_M} > 0$
and $\mathcal{E}_{-\pi_U,m} = \alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K^2}\frac{K}{\pi_U} * \mathcal{E}_{K,m} = \mathcal{E}_{K,m}\frac{\pi_U+c}{\pi_U} = -\overline{\theta}e^{-b/\mu}\mathcal{E}_{\pi_M,m} < 0$

and $\frac{d\pi_U}{dm} = -\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K^2} * -\varphi_m\frac{K_{TA}}{m} = e^{-b/\mu}\frac{d\pi_M}{dm} > 0$.
Therefore: $\mathcal{E}_{\overline{\theta},m} = \mathcal{E}_{\pi_M,m} - \mathcal{E}_{-\pi_U,m} = (1+\overline{\theta}e^{-b/\mu})\mathcal{E}_{\pi_M,m} > 0$ (by equation (27))

$\overline{\theta} = \phi^{z^*-2\overline{x}} \rightarrow \mathcal{E}_{\overline{\theta},m} = \ln\phi * (\frac{dz^*}{d\ln m} - 2\frac{d\overline{x}}{d\ln m})$
$\leftrightarrow \frac{1+\overline{\theta}e^{-b/\mu}}{2\ln\phi}\mathcal{E}_{\pi_M,m} = \frac{1}{2}z^*\mathcal{E}_{z^*,m} - \overline{x}\mathcal{E}_{\overline{x},m} < 0$ **(S1)**

Equation (28b) $\rightarrow \mathcal{E}_{\pi_M,m} = 1 - \mathcal{E}_{g(z^*,\overline{x}),m} = 1 - \frac{1}{g(z^*,\overline{x})}\frac{dg(z^*,\overline{x})}{d\ln m}$
$= 1 - \frac{1}{g(z^*,\overline{x})}([g(z^*+1,\overline{x}) - g(z^*,\overline{x})]\frac{dz^*}{d\ln m} + [g(z^*,\overline{x}) - g(z^*,\overline{x}-1)]\frac{d\overline{x}}{d\ln m})$
By Lemma A-1 on page 34:
$\mathcal{E}_{\pi_M,m} = 1 - \frac{1}{g(z^*,\overline{x})}([(\frac{1}{1+\phi}g(z^*,\overline{x}) + \frac{\phi}{1+\phi}g(z^*,\overline{x}-1)) - g(z^*,\overline{x})]\frac{dz^*}{d\ln m} + [g(z^*,\overline{x}) - g(z^*,\overline{x}-1)]\frac{d\overline{x}}{d\ln m})$
$= 1 + \ln\phi\frac{3-\overline{\theta}}{1+\overline{\theta}}(\frac{\phi}{1+\phi}\frac{dz^*}{d\ln m} - \frac{d\overline{x}}{d\ln m})$ (by Lemma A-9 on page 35)
$\leftrightarrow \frac{1+\overline{\theta}}{\ln\phi(3-\overline{\theta})}(\mathcal{E}_{\pi_M,m} - 1) = \frac{\phi}{1+\phi}z^*\mathcal{E}_{z^*,m} - \overline{x}\mathcal{E}_{\overline{x},m}$ **(S2)**

(S1) − (S2) $\leftrightarrow \frac{1+\overline{\theta}e^{-b/\mu}}{2\ln\phi}\mathcal{E}_{\pi_M,m} + \frac{1+\overline{\theta}}{\ln\phi(3-\overline{\theta})}(1-\mathcal{E}_{\pi_M,m}) = (\frac{1}{2} - \frac{\phi}{1+\phi})z^*\mathcal{E}_{z^*,m} \leftrightarrow$ Equation (39a)
$\frac{\phi}{1+\phi}$(S1) $-\frac{1}{2}$(S2) $\leftrightarrow \frac{\phi}{1+\phi}\frac{1+\overline{\theta}e^{-b/\mu}}{2\ln\phi}\mathcal{E}_{\pi_M,m} - \frac{1}{2}\frac{1+\overline{\theta}}{\ln\phi(3-\overline{\theta})}(1-\mathcal{E}_{\pi_M,m}) = (\frac{1}{2} - \frac{\phi}{1+\phi})\overline{x}\mathcal{E}_{\overline{x},m}$
$\leftrightarrow$ Equation (39b)

Note that by equations (38a) and (38b): $-\mathcal{E}_{K,m} < 1 + \mathcal{E}_{K,m} < 1 - \frac{c}{\pi_{\mathrm{M}}+c} = \frac{\pi_{\mathrm{M}}}{\pi_{\mathrm{M}}+c} \leftrightarrow$
$1 > -\mathcal{E}_{K,m}\frac{\pi_M+c}{\pi_M} = \mathcal{E}_{\pi_M,m}$ □

Proof of equation (40)*:* $\mathcal{E}_{v^*,m} = \mathcal{E}_{v,\overline{x}}(\mathcal{E}_{\overline{x},m} - \mathcal{E}_{\overline{x},z}\mathcal{E}_{z^*,m}) = \frac{\partial v}{\partial\overline{x}}\frac{\overline{x}}{v^*}(\mathcal{E}_{\overline{x},m} - \frac{\partial\overline{x}}{\partial z}\frac{z^*}{\overline{x}}\mathcal{E}_{z^*,m})$
$= \frac{1+\overline{\theta}}{2}g(z,\overline{x})\frac{\overline{x}}{v^*}(\frac{\frac{\phi}{1+\phi}\frac{1+\overline{\theta}e^{-b/\mu}}{2}\mathcal{E}_{\pi_M,m} + \frac{1}{2}\frac{1+\overline{\theta}}{3-\overline{\theta}}(1-\mathcal{E}_{\pi_M,m})}{\overline{x}\ln\phi(\frac{1}{2}-\frac{\phi}{1+\phi})} - \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})}\frac{z^*}{\overline{x}}\frac{\frac{1+\overline{\theta}e^{-b/\mu}}{2}\mathcal{E}_{\pi_M,m} + \frac{1+\overline{\theta}}{3-\overline{\theta}}(1-\mathcal{E}_{\pi_M,m})}{z^*\ln\phi(\frac{1}{2}-\frac{\phi}{1+\phi})})$
(By equations (14), (39b), (16) , and (39a))
$\leftrightarrow$ Equation (40).

Note: $\frac{\phi}{1+\phi} - \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})} = \frac{\phi(1+\overline{\theta})-(\phi+\overline{\theta})}{(1+\phi)(1+\overline{\theta})} = -\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}} < 0$
and $\overline{\theta} \lesseqgtr 1 \leftrightarrow (1-\phi)(1-\overline{\theta}) \gtreqless 0 \leftrightarrow (1+\phi)(1+\overline{\theta}) \gtreqless 2(\phi+\overline{\theta}) \leftrightarrow \frac{1}{2} \gtreqless \frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})}$ □

Proof of equations (41) *and* (42)*:* If targeted advertising equilibrium:
Equation (30) $\leftrightarrow n_{TA} = \frac{\mu}{2TC(z^*,v^*)} - \frac{\mu/K}{2\alpha\mu v^*[\frac{1}{1+\theta^*}e^{\frac{R+b-\mu}{\mu}} + \frac{\theta^*}{1+\theta^*}e^{\frac{R-\mu}{\mu}}]/K}$
$\leftrightarrow$ Equation (41) (By the zero-profit condition)

If mass advertising equilibrium:
Equation (20) $\leftrightarrow n_{MA} = \frac{\mu}{2TC(z^*=0,v^*=1)} - \frac{\mu/K}{2\alpha\mu[e^{\frac{R+b-\mu}{\mu}} + e^{\frac{R-\mu}{\mu}}]/K}$
$\leftrightarrow$ Equation (41) (By the zero-profit condition)

If mixed advertising equilibrium:
Equation (34) $\leftrightarrow (F+c)n_{MA} = (F+c)\overline{n}_{MA} + (F+mz^*+cv^*)\overline{n}_{TA} = (F+mz^*+cv^*)n_{TA}$
$\leftrightarrow$ Equation (42) (By what I just found for targeted and mass advertising equilibria) □

Appendix B. Additional Proofs

Lemma A-1. $g(x,z) = \frac{\phi}{1+\phi}g(x-1,z-1) + \frac{1}{1+\phi}g(x,z-1)$

Proof: $g(x,z) = \binom{z}{x}\left(\frac{1}{1+\phi}\right)^{z-x}\left(\frac{\phi}{1+\phi}\right)^{x} = [\binom{z-1}{x-1} + \binom{z-1}{x}]\left(\frac{1}{1+\phi}\right)^{z-x}\left(\frac{\phi}{1+\phi}\right)^{x}$
$= \frac{\phi}{1+\phi} * \binom{z-1}{x-1}\left(\frac{1}{1+\phi}\right)^{(z-1)-(x-1)}\left(\frac{\phi}{1+\phi}\right)^{x-1} + \frac{1}{1+\phi} * \binom{z-1}{x}\left(\frac{1}{1+\phi}\right)^{(z-1)-x}\left(\frac{\phi}{1+\phi}\right)^{x}$
$= \frac{\phi}{1+\phi}g(x-1,z-1) + \frac{1}{1+\phi}g(x,z-1)$ □

Lemma A-2. $g(x,z) - g(x-1,z) = \frac{(z+1)-x/\frac{\phi}{1+\phi}}{(z+1)-x}g(x,z)$

Proof: $g(x,z) = \binom{z}{x}\left(\frac{1}{1+\phi}\right)^{z-x}\left(\frac{\phi}{1+\phi}\right)^{x}$
$\leftrightarrow g(x-1,z) = \binom{z}{x-1}\left(\frac{1}{1+\phi}\right)^{z-(x-1)}\left(\frac{\phi}{1+\phi}\right)^{x-1} = \frac{1}{\phi}\frac{x}{z-(x-1)}g(x,z)$
$\leftrightarrow g(x,z) - g(x-1,z) = [1 - \frac{1}{\phi}\frac{x}{z-(x-1)}]g(x,z) = \frac{(z+1)-x/\frac{\phi}{1+\phi}}{(z+1)-x}g(x,z)$ □

Lemma A-3. $\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}(\overline{\theta}(z,\overline{x})g(z,\overline{x}) - \overline{\theta}(z,\overline{x}-1)g(z,\overline{x}-1))$

Proof: Define $\hat{g}(z,x) \equiv \theta(z,x)g(z,x) = \Pr(x_{ij} = x|\tau_i = 1-t_j)$.
In the proof of Lemma 6, I showed that:
$\frac{d_2\Pi}{dz^2} = \left([\frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}] + 2\frac{1+\overline{\theta}}{2}[\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} - c]\right)\frac{d}{dz}g(z,\overline{x}(z,v))$
$= \left([\frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{1}{1+\overline{\theta}}\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}] + 2\frac{1+\overline{\theta}}{2\overline{\theta}}[\alpha\mu\frac{\frac{1}{1+\overline{\theta}}e^{\frac{R+b-\mu}{\mu}} + \frac{\overline{\theta}}{1+\overline{\theta}}e^{\frac{R-\mu}{\mu}}}{K} - c]\right)\frac{d}{dz}\hat{g}(z,\overline{x}(z,v))$
$\rightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}(\frac{m\frac{d}{dz}\hat{g}(z,\overline{x}(z,v))}{\hat{g}(z,\overline{x}(z,v))})$ (because $\frac{d\Pi}{dz} = 0$ and $\frac{d\Pi}{dv} = 0$ in equilibrium)
$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}(\frac{d}{dz}\hat{g}(z,\overline{x}(z,v)))$ (because $m > 0$ and $\hat{g}(z,\overline{x}(z,v)) > 0$)
$\leftrightarrow$ Because $z - 2\overline{x}$ is constant:
$\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}([\hat{g}(z+1,\overline{x}) - \hat{g}(z,\overline{x})] + 2[\hat{g}(z,\overline{x}) - \hat{g}(z,\overline{x}-1)])$
$\leftrightarrow$ By Lemma A-1 on pg. 34:
$\text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}([(\frac{1}{1+\phi}\hat{g}(z,\overline{x}-1) + \frac{\phi}{1+\phi}\hat{g}(z,\overline{x})) - \hat{g}(z,\overline{x})] + 2[\hat{g}(z,\overline{x}) - \hat{g}(z,\overline{x}-1)])$
$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}((2 - \frac{1}{1+\phi})[\hat{g}(z,\overline{x}) - \hat{g}(z,\overline{x}-1)])$
$\leftrightarrow \text{sgn}(\frac{d_2\Pi}{dz^2}(z^*,v^*)) = \text{sgn}(\hat{g}(z,\overline{x}) - \hat{g}(z,\overline{x}-1))$ (because $2 > \frac{1}{1+\phi}$) □

Lemma 9. *I have that:*

(a) $h(z,v) \equiv \frac{g(z,\overline{x}(z,v))-g(z,\overline{x}(z,v)-1)}{g(z,\overline{x}(z,v))} = \ln\phi(\frac{1-\overline{\theta}}{1+\overline{\theta}}\frac{\tilde{m}}{MR_z} + \frac{2}{1+\overline{\theta}})$

(b) $h^* \equiv h(z^*,v^*) = \ln\phi\frac{3-\overline{\theta}}{1+\overline{\theta}} < 0$

(c) $\overline{\theta}(z^*,v^*) \in \{0,3\}$

where $\tilde{m} \equiv \frac{1}{2}(\frac{1-\phi}{1+\phi})g(z,\overline{x}(z,v))\tilde{\pi}_M$.

Proof: Because $\overline{\theta} = \phi^{z-2\overline{x}}$:

$\frac{\partial\overline{\theta}}{\partial v} = \overline{\theta} * -2\ln\phi * \frac{\partial\overline{x}}{\partial v} = \frac{-2\overline{\theta}\ln\phi}{\frac{1+\overline{\theta}}{2}g(z,\overline{x})} > 0$ (by equation (14))

$\frac{\partial\overline{\theta}}{\partial z} = \overline{\theta} * \ln\phi * [1 - 2\frac{\partial\overline{x}}{\partial z}] = \overline{\theta}\ln\phi[1 - 2\frac{\phi+\overline{\theta}}{(1+\phi)(1+\overline{\theta})}] = \overline{\theta}\ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{1+\overline{\theta}}$ (by equation (16))

Equation (23) $\rightarrow \frac{\partial}{\partial z}MR_v = -\alpha\mu\frac{\frac{1}{(1+\overline{\theta})^2}(e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}})}{K} * \overline{\theta}\ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{1+\overline{\theta}}$

$= \alpha\mu\frac{\frac{1}{(1+\overline{\theta})}(e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}})}{K} * -\overline{\theta}\ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{(1+\overline{\theta})^2} = -\overline{\theta}[\alpha\mu\frac{e^{\frac{R-\mu}{\mu}}}{K} - MR_v] * \ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{(1+\overline{\theta})^2}$

$= \ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{(1+\overline{\theta})^2} * \tilde{\pi}_M$

Note that $\tilde{\pi}_M = \pi_M$ in equilibrium by the first-order condition of v and equation (27).

Equation (24) $\rightarrow$

$\frac{\partial}{\partial v}MR_z = \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{1}{(1+\overline{\theta})^2}g(z,\overline{x})\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K} * \frac{-2\overline{\theta}\ln\phi}{\frac{1+\overline{\theta}}{2}g(z,\overline{x})} + \frac{\alpha\mu}{2}\frac{1-\phi}{1+\phi}\frac{\overline{\theta}}{1+\overline{\theta}}\frac{\partial}{\partial v}g(z,\overline{x}(z,v))\frac{e^{\frac{R+b-\mu}{\mu}} - e^{\frac{R-\mu}{\mu}}}{K}$

$= MR_z[\frac{-2\ln\phi}{\frac{(1+\overline{\theta})^2}{2}g(z,\overline{x})} + h * \frac{\partial\overline{x}}{\partial v}] = \frac{MR_z}{\frac{1+\overline{\theta}}{2}g(z,\overline{x})}[\frac{-2\ln\phi}{1+\overline{\theta}} + h]$ (by equation (14))

By Young's theorem $\frac{\partial}{\partial z}MR_v = \frac{\partial}{\partial v}MR_z \leftrightarrow \ln\phi\frac{1-\phi}{1+\phi}\frac{1-\overline{\theta}}{(1+\overline{\theta})^2} * \tilde{\pi}_M = \frac{MR_z}{\frac{1+\overline{\theta}}{2}g(z,\overline{x})}[\frac{-2\ln\phi}{1+\overline{\theta}} + h]$

$= \ln\phi\frac{1-\overline{\theta}}{1+\overline{\theta}}\frac{\tilde{m}}{MR_z} = \frac{-2\ln\phi}{1+\overline{\theta}} + h$

Note that $\tilde{m} = m$ and therefore MR_z in equilibrium by equation (28b).

$\leftrightarrow h = \ln\phi(\frac{1-\overline{\theta}}{1+\overline{\theta}}\frac{\tilde{m}}{MR_z} + \frac{2}{1+\overline{\theta}})$. In equilibrium $\tilde{m} = MR_z \rightarrow h^* = \ln\phi\frac{3-\overline{\theta}}{1+\overline{\theta}}$, which has to be less than zero by Lemma 7. □

Lemma A-4. $\frac{d^2K_{TA}}{dm^2} > 0$

Proof: Currier (2002, p. 79-82)

Silberberg's (1974) primal-dual formulation of the dual problem given by equations (36) and (29) would be:

$$\max_{z,v,K,F,m,c} K - K(F,m,c)$$
$$\text{subject to } TR(z,v,K) - [F + mz^* + cv^*] = 0$$

where $K(F,m,c)$ would be the optimal value of K as a function of F, m, and c. Note that for a fixed K, z, and v, that this problem is the same as minimizing $K(F,m,c)$ subject to a linear constraint. Because $\frac{dK}{dm} < 0$ and in a similar fashion $\frac{dK}{dc} < 0$ and

$\frac{dK}{dF} < 0$, $K(F, m, c)$ must be locally convex in (F, m, c) at the equilibrium values of z^* and v^*. □

Appendix C. Examination of equations (34) and (35)

During the special case where $K_{MA} = K_{TA}$, then and only then can I have a mixture of mass advertisers and targeted advertisers. Equations (34) and (35) define the sufficient and necessary conditions for the number of each type of advertiser in each sub-market. It describes the trade-offs between mass and targeted advertiser in each sub-market and between each sub-market. It is straightforward to show that the slopes of these trade-offs are not equal.

Lemma B-1. $2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1} < 2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}$

Proof: $\theta^* < 1 \leftrightarrow 2e^{b/\mu} \frac{\theta^*}{1+\theta^*} < 2e^{b/\mu} \frac{1}{1+\theta^*}$
$\leftrightarrow (e^{b/\mu} - 1)(\frac{1}{1+\theta^*} e^{b/\mu} + \frac{\theta^*}{1+\theta^*}) < (e^{b/\mu} + 1)(\frac{1}{1+\theta^*} e^{b/\mu} - \frac{\theta^*}{1+\theta^*}) \leftrightarrow$ The Lemma □

From this, I have that the number of each type of advertiser in each sub-market can be represented by Figure C.1(a). There is a line between $(n_{MA}, 0)$ and $(0, n_{TA})$, defined by equation (34). This line has a slope of $-2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}$. The point $(\overline{n}_{MA}, \overline{n}_{TA})$, which represents the average number of each type of advertiser in each sub-market, must be on this line. How much (n^0_{MA}, n^0_{TA}) and (n^1_{MA}, n^1_{TA}) deviate from $(\overline{n}_{MA}, \overline{n}_{TA})$ is defined by equation (35). This defines a line (given by the dashed line) that runs through $(\overline{n}_{MA}, \overline{n}_{TA})$ with a slope of $-2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}$. By Lemma B-1, the slope of this line must be steeper than the slope of the line defined by equation (34).

This creates a trapezoidal range of possible (n^0_{MA}, n^0_{TA}) and (n^1_{MA}, n^1_{TA}) shown in Figure C.1(b). Given (n^0_{MA}, n^0_{TA}), there is one unique (n^1_{MA}, n^1_{TA}) (and visa versa). Both points must be on the same line with a slope of $-2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}$, given by the dashed lines. And both points must be equidistant from the solid black line through $(n_{MA}, 0)$ and $(0, n_{TA})$.

One extreme combination is $(2\widetilde{n}_{MA}, 0)$ and $(0, 2\widetilde{n}_{TA})$. where all entrants in one sub-market mass advertise and all entrants in the other market target advertise.

$$\widetilde{n}_{TA} = \frac{e^{2b/\mu} - 1}{4v^*[\frac{1}{1+\theta^*} e^{2b/\mu} - \frac{\theta^*}{1+\theta^*}]} n_{MA} \tag{C.1}$$

$$\widetilde{n}_{MA} = 2v^* \frac{\frac{1}{1+\theta^*} e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu} - 1} \widetilde{n}_{TA} > \frac{n_{MA}}{2} \tag{C.2}$$

Proof: From equation (34), I have: $n_{MA} = \widetilde{n}_{MA} + 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\widetilde{n}_{TA}$.

From equation (35), I have: $-\frac{\widetilde{n}_{MA}}{\widetilde{n}_{TA}} = -2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1} \leftrightarrow \widetilde{n}_{MA} = 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}\widetilde{n}_{TA}$.

Therefore: $n_{MA} = 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}\widetilde{n}_{TA} + 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}\widetilde{n}_{TA} = 4v^* \frac{\frac{1}{1+\theta^*}e^{2b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{2b/\mu}-1}\widetilde{n}_{TA}$

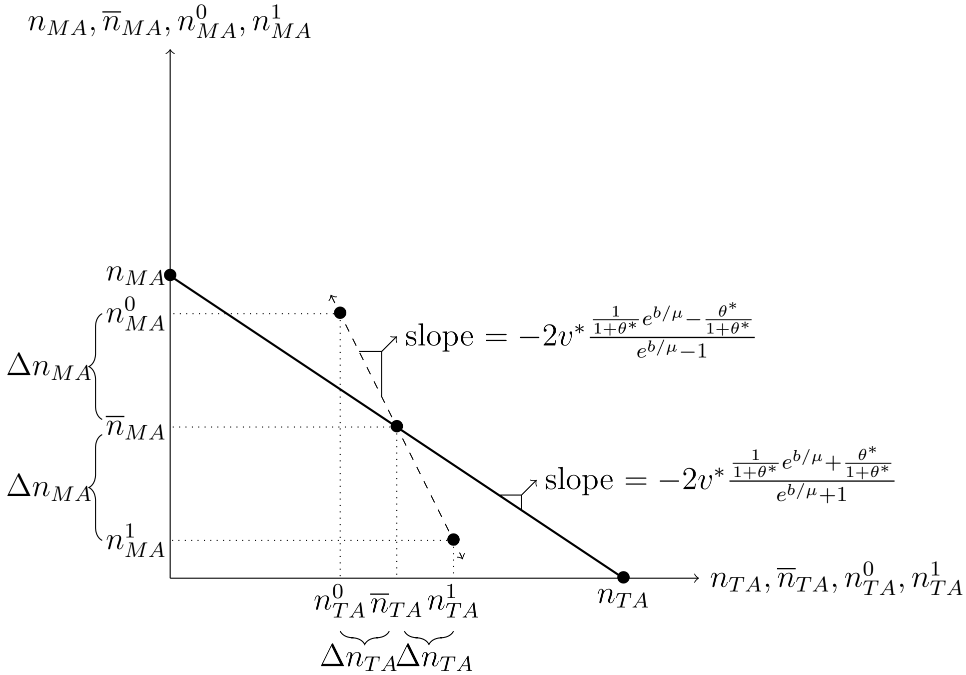
$\leftrightarrow$ Equation (C.1)

$\leftrightarrow \widetilde{n}_{MA} = 2v^* \frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1} * \widetilde{n}_{TA}$

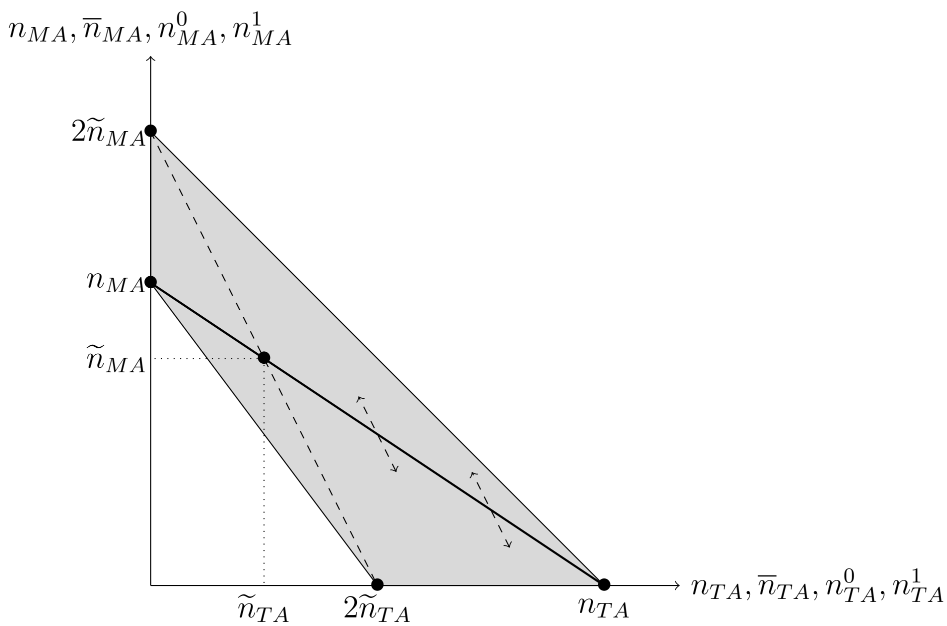
$$= \frac{\frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1}}{\frac{\frac{1}{1+\theta^*}e^{b/\mu} - \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}-1} + \frac{\frac{1}{1+\theta^*}e^{b/\mu} + \frac{\theta^*}{1+\theta^*}}{e^{b/\mu}+1}} n_{MA} > \frac{n_{MA}}{2} \qquad \text{(by Lemma B-1)}$$

□

Figure C.1: Possible Mixed Advertising Equilibrium Number of Entrants of Each Type



(a) One Possible Equilibrium



(b) Range of Possible Equilibria